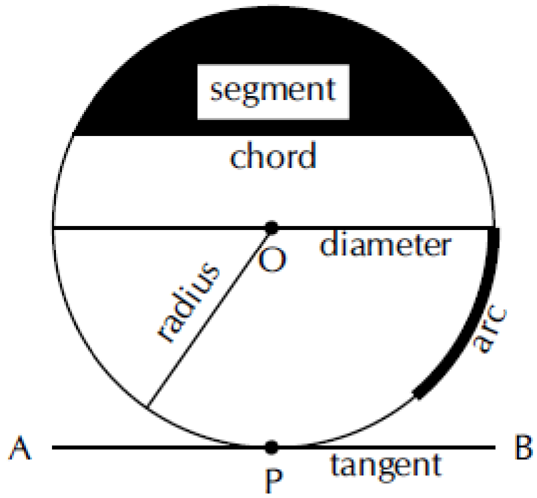
GRADE 11 EUCLIDEAN GEOMETRY

4. CIRCLES

4.1 TERMINOLOGY



Arc	An arc is a part of the circumference of a circle
Chord	A chord is a straight line joining the ends of an arc.
Radius	A radius is any straight line from the centre of the circle to a point on the circumference
Diameter	A diameter is a special chord that passes through the centre of the circle. A Diameter is the length of a straight line segment from one point on the circumference to another point on the circumference, that passes through the centre of the circle.
Segment	A segment is the part of the circle that is cut off by a chord. A chord divides a circle into two segments
Tangent	A tangent is a line that makes contact with a circle at one point on the circumference (AB is a tangent to the circle at point P).

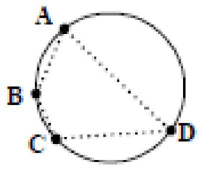
4.2 SUMMARY OF THEOREMS

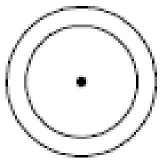
4.2.1 Definitions

* Points are concyclic if they lie on the circumference of a circle.

* A quadrilateral is cyclic if all four vertices lie on the circumference of a circle.

* Concentric circles have the same centre.

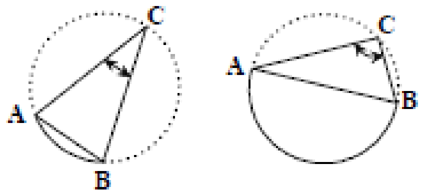




Concentric circles

Concyclic points
[$ABCD$ is a cyclic quadrilateral]

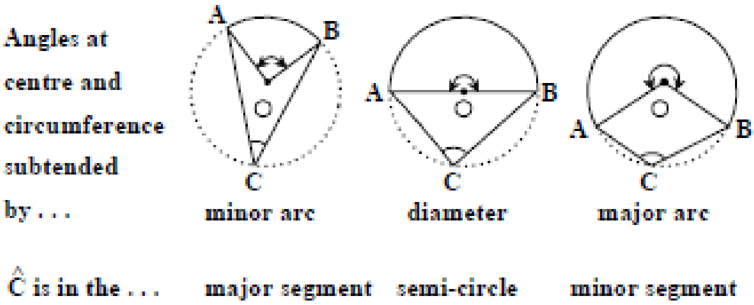
* An arc (or chord) of a circle subtends an angle if the arms of the angle are joined by the arc (or chord)



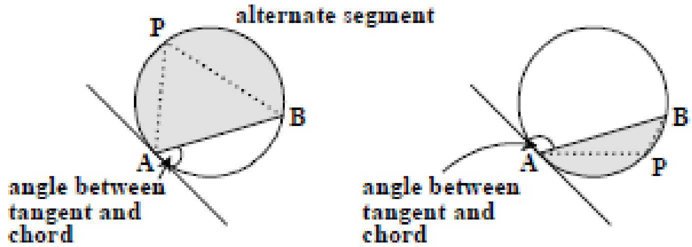
Arc (chord) AB subtends $\hat{C}$

* An angle is at the centre when its arms are radii.

* An angle is at the circumference (or in a segment) of a circle when its arms are chords.



* The chord AB subtends angle P in the segment opposite to the selected angle between the tangent and chord AB.



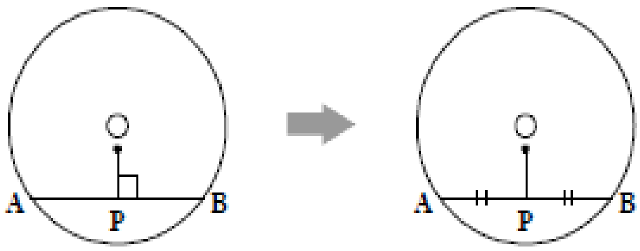
4.2.2 Chords and Midpoints

Note: O indicates the centre of the circle.

- **Theorem 1:**

A line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.

($OP \perp AB$)

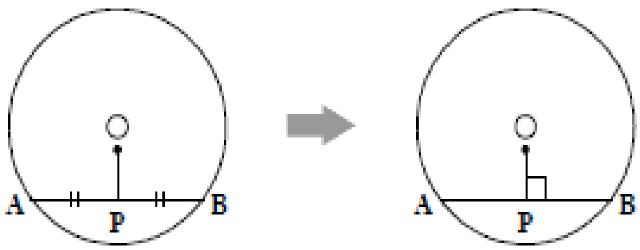


if $OP \perp AB$ **then** $AP = PB$

- **Theorem 2:**

(Converse of theorem 1).
The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

($AP = PB$)

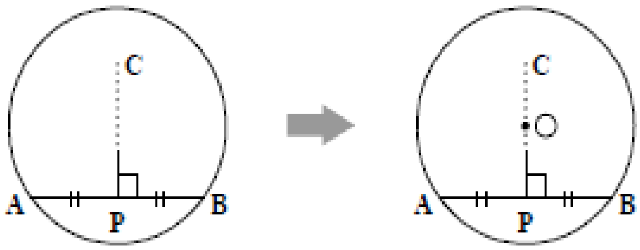


if $AP = PB$ **then** $OP \perp AB$

- **Theorem 3:**

The perpendicular bisector of a chord passes through the centre of the circle.

($AP = PB$ and $CP \perp AB$)



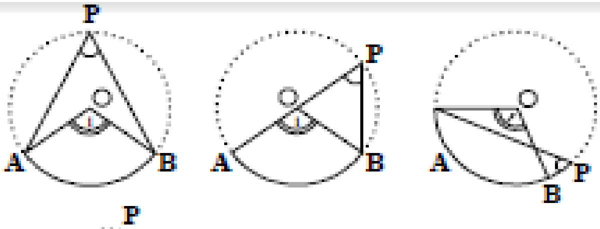
$$\text{if} \begin{bmatrix} AP = PB \\ \text{and} \\ CP \perp AB \end{bmatrix} \text{then} \begin{bmatrix} PC \text{ passes} \\ \text{throught } O \end{bmatrix}$$

4.2.3 Angles in circles

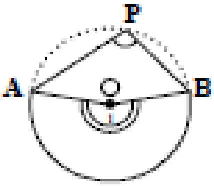
- **Theorem 4:**

The angle at the centre is twice the angle at the circumference subtended by the same arc.

($\angle$ at centre = $2 \times \angle$ at circumf.)



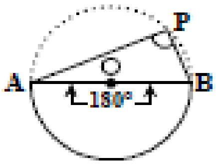
$\hat{O}_1 = 2\hat{P}$ subtended by minor arc AB



$\hat{O}_1 = 2\hat{P}$ subtended by major arc AB

* **Important deductions:**

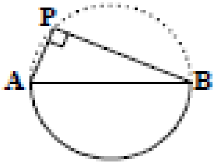
1. An angle in a semi-circle is a right angle



$2\hat{P} = A\hat{O}B = 180°$

$\therefore \hat{P} = 90°$

2. The chord that subtends a right angle at the circumference is a diameter.



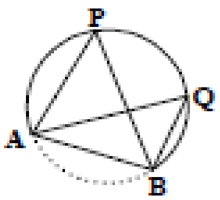
if $\hat{P} = 90°$ **then** AB is a diameter

- **Theorem 5:**

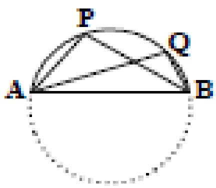
Angles subtended by a chord at the circumference of a circle, on the same side of the chord, are equal.
(In other words: Angles in the same segment of a circle are equal.)

($\angle s$ in same segment)

if



or



then $\hat{P} = \hat{Q}$

[$\hat{P}$ and $\hat{Q}$ are in the major segment]

[$\hat{P}$ and $\hat{Q}$ are in the minor segment]

- **Theorem 6:**

(Converse of theorem 5).
If a line segment joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.

(AB subtends equal $\angle s$)

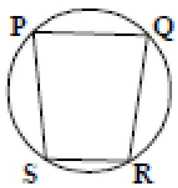
if $\hat{P} = \hat{Q}$ **then** A, B, P **and** Q are concyclic
[i.e. $ABPQ$ is a cyclic quadrilateral]

4.2.4 Cyclic Quadrilaterals

- **Theorem 7:**

The opposite angles of a cyclic quadrilateral are supplementary.
(opp. ∠s of cyc. quad.)

if

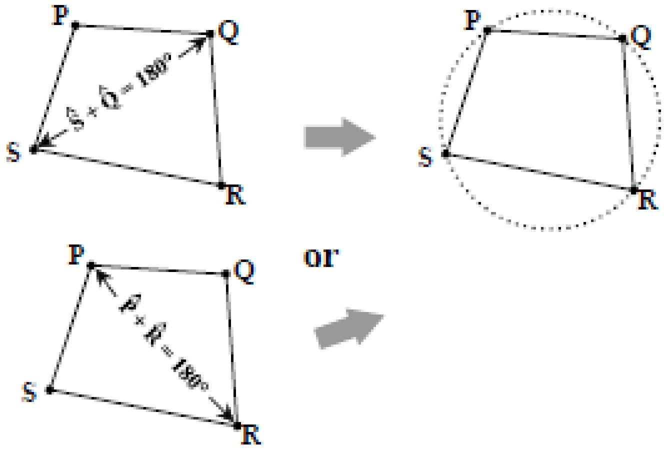


then $\left[\begin{matrix}\hat{P}+\hat{R}=180^\circ \\ \textbf{and} \\ \hat{Q}+\hat{A}=180^\circ\end{matrix}\right]$

- **Theorem 8:**

(Converse of theorem 7). If two opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

(opp. ∠s supp.)

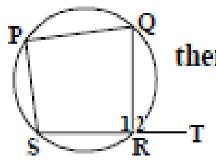


if $\left[\begin{matrix}\hat{S}+\hat{Q}=180^\circ \\ \textbf{or} \\ \hat{P}+\hat{R}=180^\circ\end{matrix}\right]$ **then** *PQRS* is a cyclic quadrilateral

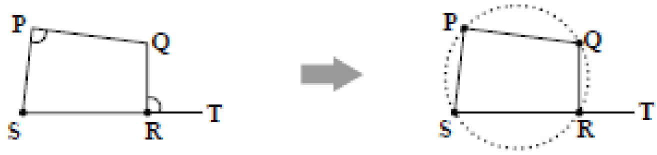
* **Important deductions:**

1. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
   (ext. ∠ of cyc. quad.)
2. (Converse of deduction 1). If the exterior angle of a quadrilateral is equal to the interior opposite angle, the quadrilateral is cyclic.
   (ext. ∠ = opp. int. ∠)

if



then $\hat{R}_2 = \hat{P}$

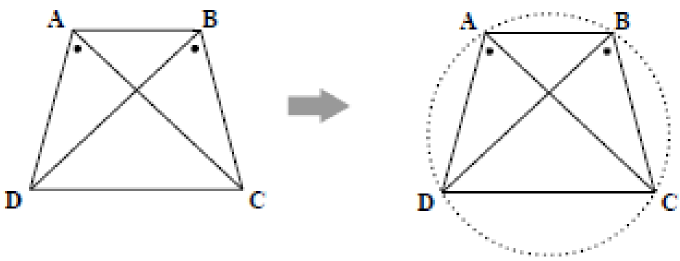


if $Q\hat{R}T = \hat{P}$ **then** *PQRS* is a cyclic quadrilateral.

* **Important reminder:**

1. A third way of proving that a quadrilateral is cyclic, is by using theorem 6.

2. Once you have proven a quadrilateral to be cyclic, draw a light circle around it for further use.



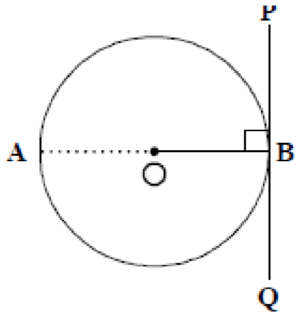
if $\hat{A}=\hat{B}$ **then** *ABCD* is a cyclic quadrilateral.

4.2.4 Tangents

- **Axiom:**

A tangent is perpendicular to the radius (or diameter) at the point of contact.

(radius OB $\perp$ tangent PQ)

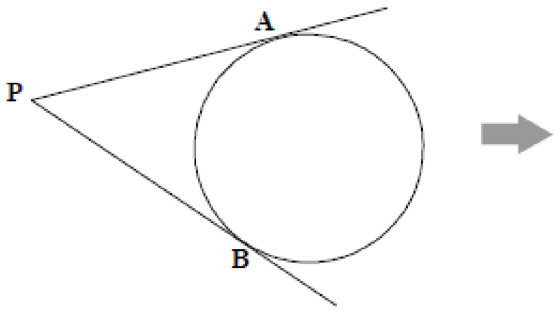


tangent $PBQ \perp$ radius OB

- **Theorem 9:**

Two tangents drawn to a circle from the same point outside the circle are equal in length.

(tangents from same point)



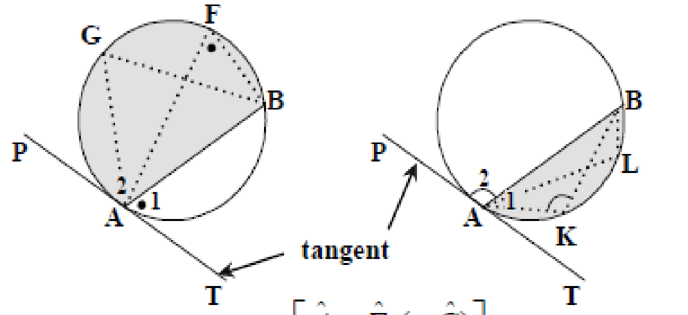
PA = PB

if PA and PB are tangents **then** $PA = PB$

- **Theorem 10:**

The angle between a tangent and a chord drawn to the point of contact is equal to the angles in the alternate segment.

(∠ between tangent and chord)

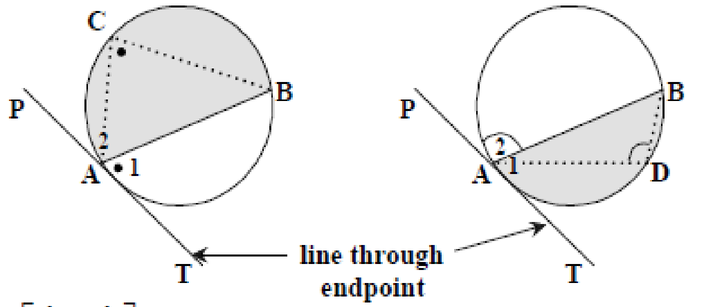


if PT is a tangent **then** $\left[\begin{array}{c}\hat{A}_1 = \hat{F}\ (= \hat{G}) \\ \textbf{and} \\ \hat{A}_2 = \hat{K}\ (= \hat{L})\end{array}\right]$

- **Theorem 11:**

(Converse of theorem 10).
If the line through the endpoint of a chord makes an angle with the chord, equal to an angle in the alternate segment, then the line is a tangent to the circle.

(∠ between line and chord = ∠ in opp. segm.)



if $\left[\begin{array}{c}\hat{A}_1 = \hat{C} \\ \textbf{or} \\ \hat{A}_2 = \hat{D}\end{array}\right]$ **then** PT is a tangent

4.3 PROOF OF THEOREMS

All SEVEN theorems listed in the CAPS document must be proved. However, there are four theorems whose proofs are examinable (according to the Examination Guidelines 2014) in grade 12. In this guide, only FOUR examinable theorems are proved. These **four** theorems are written in **bold.**

1. **The line drawn from the centre of a circle perpendicular to the chord bisects the chord.**

2. The perpendicular bisector of a chord passes through the centre of the circle.

3. **The angle subtended by an arc at the centre of a circle is double the angle subtended by the same arc at the circle (on the same side of the arc as the centre).**

4. Angles subtended by an arc or chord of the circle on the same side of the chord are equal.

5. **The opposite angles of a cyclic quadrilateral are supplementary.**

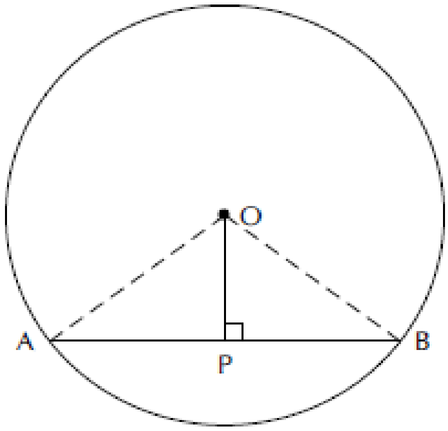
6. Two tangents drawn to a circle from the same point outside the circle are equal in length (If two tangents to a circle are drawn from a point outside the circle, the distances between this point and the points of contact are equal).

7. **The angle between the tangent of a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.**

The above theorems and their converses, where they exist, are used to prove riders.

Theorem 1. *The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.*

Proof:



Consider a circle, with centre O. Draw a chord AB and draw a perpendicular line from the centre of the circle to intersect the chord at point P.

The aim is to prove that $AP = BP$

1. $\triangle OAP$ and $\triangle OBP$ are right-angled triangles.
2. $OA = OB$ as both of these are radii and OP is common to both triangles.

Apply the Theorem of Pythagoras to each triangle, to get:

$$\begin{aligned} OA^2 &= OP^2 + AP^2 \\ OB^2 &= OP^2 + BP^2 \end{aligned}$$

However, $OA = OB$. So,

$$\begin{aligned} OP^2 + AP^2 &= OP^2 + BP^2 \\ \therefore AP^2 &= BP^2 \\ \text{and } AP &= BP \end{aligned}$$

This means that OP bisects AB.

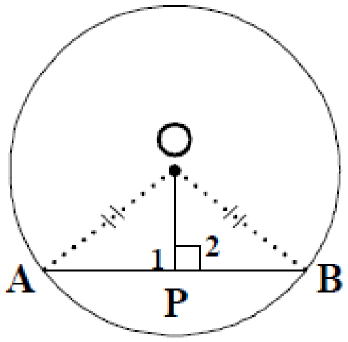
OR

Theorem 1

The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.

Given:
Circle with centre O and chord $AB \perp OP$

To Prove:
$AP = PB$



Proof:

Draw OA and OB

In ΔOAP and ΔOBP

$OA = OB$ (radii)

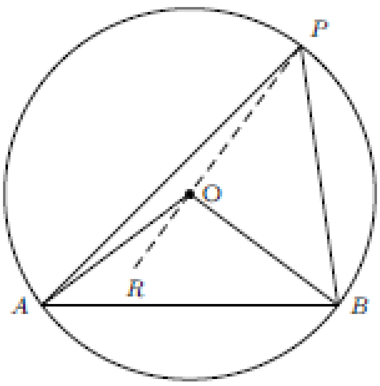
$OP = OP$ (common)

$\hat{P}_1 = \hat{P}_2 = 90°$ (given)

$\therefore \Delta OAP \equiv \Delta OBP$ $(90°, h, s)$

$\therefore \quad AP = BP$

Theorem 4. *The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.*



Consider a circle, with centre O and with A and B on the circumference. Draw a chord AB. Draw radii OA and OB. Select any point P on the circumference of the circle. Draw lines PA and PB. Draw PO and extend to R.

The aim is to prove that $A\hat{O}B = 2\,.\,A\hat{P}B$.

$A\hat{O}R = P\hat{A}O + A\hat{P}O$ (exterior angle = sum of interior opp. angles)

But, $P\hat{A}O = A\hat{P}O$ ($\triangle AOP$ is an isosceles $\triangle$)

$\therefore A\hat{O}R = 2A\hat{P}O$

Similarly, $B\hat{O}R = 2B\hat{P}O$.

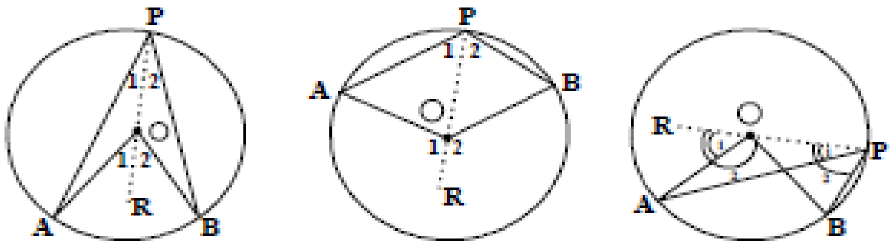
So,

$$\begin{aligned} A\hat{O}B &= A\hat{O}R + B\hat{O}R \\ &= 2A\hat{P}O + 2B\hat{P}O \\ &= 2(A\hat{P}O + B\hat{P}O) \\ &= 2(A\hat{P}B) \end{aligned}$$

OR

Theorem 4

The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.



(i) (ii) (iii)

Given:

Circle centre O and arc AB subtending $A\hat{O}B$ at the centre and $A\hat{P}B$ at the circumference.

To Prove:

$A\hat{O}B = 2A\hat{P}B$

Proof:

Join PO and produce to R

$AO = PO$ (radii)

$\therefore \hat{A} = \hat{P}_1$ ($\angle s$ opposite equal sides)

$\hat{O}_1 = \hat{A} + \hat{P}_1$ (exterior $\angle$ of ΔAPO)

$\therefore \hat{O}_1 = 2\hat{P}_1$

Similarly $\hat{O}_2 = 2\hat{P}_2$

In figures (i) and (ii)

$\hat{O}_1 + \hat{O}_2 = 2\hat{P}_1 + 2\hat{P}_2$

$= 2(\hat{P}_1 + \hat{P}_2)$

$\therefore A\hat{O}B = 2A\hat{P}B$

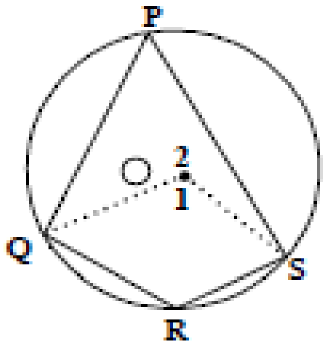
In figure (iii)

$\hat{O}_2 - \hat{O}_1 = 2\hat{P}_2 - 2\hat{P}_1$

$= 2(\hat{P}_2 - \hat{P}_1)$

$\therefore A\hat{O}B = 2A\hat{P}B$

Theorem 7
The opposite angles of a cyclic quadrilateral are supplementary.

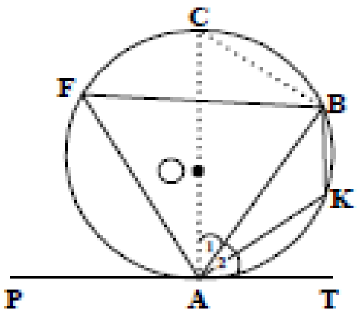


Given:
Cyclic quadrilateral $PQRS$
To Prove:
$\hat{P}+\hat{R}=180°$ and
$\hat{Q}+\hat{S}=180°$
Proof:
Join OQ and OS
$\hat{O}_1=2\hat{P}$ (∠ at centre = 2×∠ at circumference)
$\hat{O}_2=2\hat{R}$ (∠at centre=2×∠ at circumference)
$\therefore \quad \hat{O}_1+\hat{O}_2=2(\hat{P}+\hat{R})$
But $\hat{O}_1+\hat{O}_2=360°$ (∠s around point)
$\therefore \quad \hat{P}+\hat{R}=180°$
Similarly by drawing OP and OR
$\hat{Q}+\hat{S}=180°$

Theorem 10

The angle between a tangent and a chord, drawn at the point of contact, is equal to the angle which the chord subtends in the alternate segment.

Given:
Tangent PT touching circle O at A; chord AB with $\hat{F}$ in the major segment and $\hat{K}$ in the minor segment.



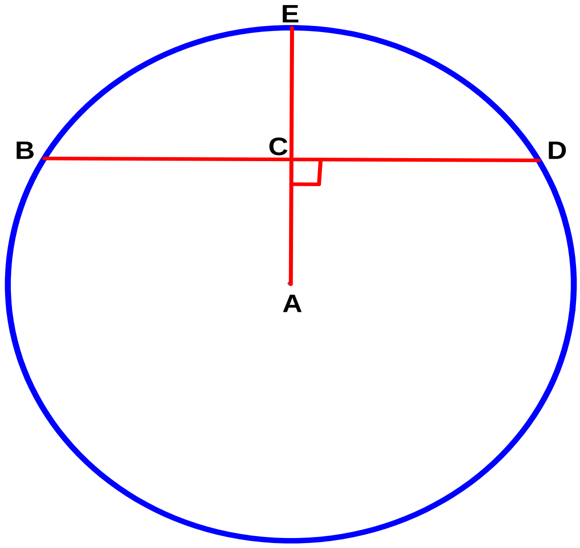
To Prove:
(i) $B\hat{A}T = \hat{F}$
(ii) $P\hat{A}B = \hat{K}$

Proof:
Draw diameter AC.
Join BC.
$\hat{A}_1 + \hat{A}_2 = 90°$ (radius $\perp$ tangent)
But $C\hat{B}A = 90°$ ($\angle$ in semi-circle)
$\therefore \quad \hat{A}_1 + \hat{C} = 90°$ ($\angle$s of ΔABC)
$\therefore \quad \hat{A}_2 = \hat{C}$
But $\hat{C} = \hat{F}$ (subtended by AB)
$\therefore \quad \hat{A}_2 = \hat{F}$
$\therefore (i)\ B\hat{A}T = \hat{F}$
$P\hat{A}B + \hat{A}_2 = 180°$ (adjacent $\angle$s on straight line)
and $\hat{K} + \hat{F} = 180°$ (opposite $\angle$s of cyclic quadrilateral)
But $\quad \hat{F} = \hat{A}_2$ (proved in (i))
$\therefore (ii)\ P\hat{A}B = \hat{K}$

4.4 ACTIVITIES

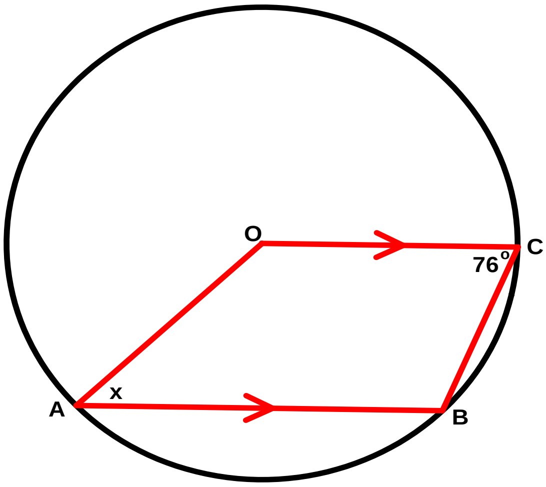
GEOMETRY 1

1. The sketch alongside shows chord BD cutting AE at C. A is the centre of the circle and AE ⊥ BD. If EC = 3cm and BD = 14cm, calculate the area of the circle.

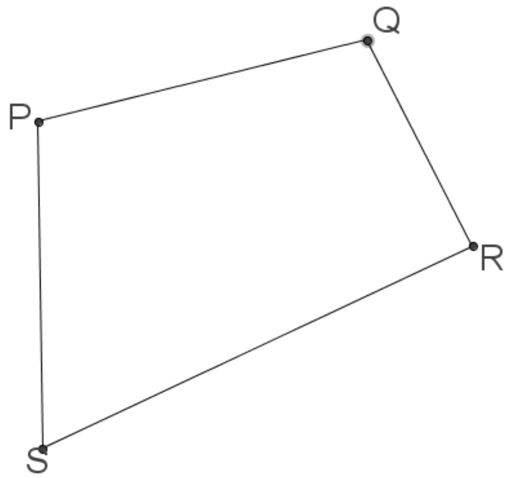
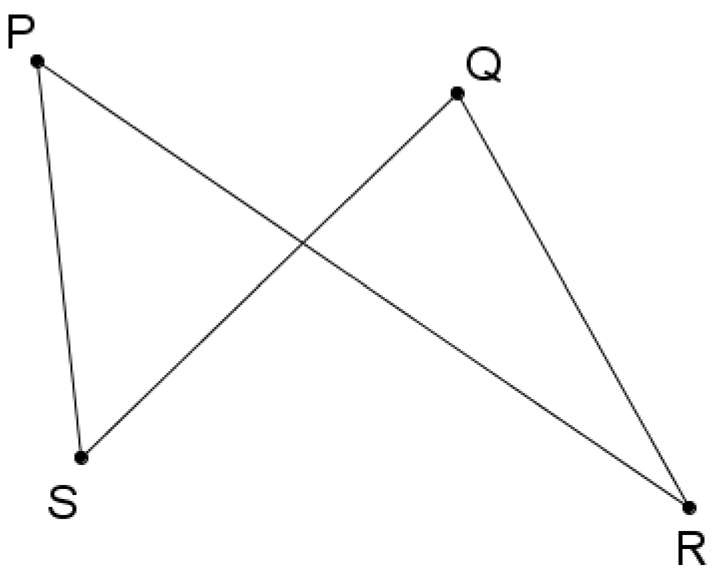
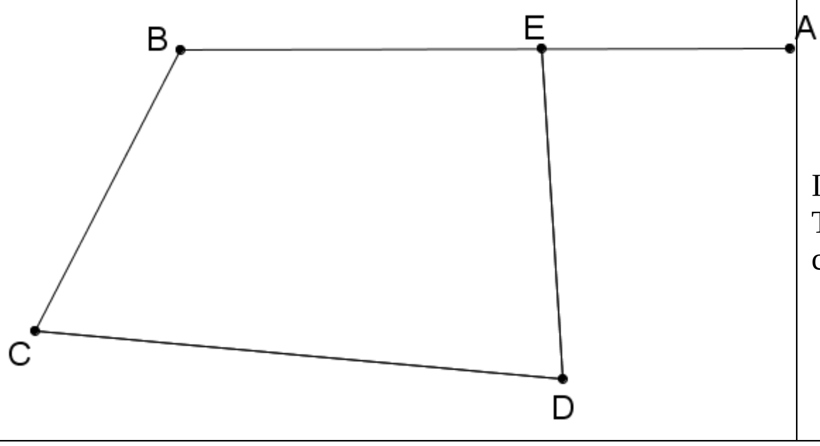


2. The sketch shows circle centre O with OC ∥ AB . $O\hat{C}B$ = 76º and Â = x.

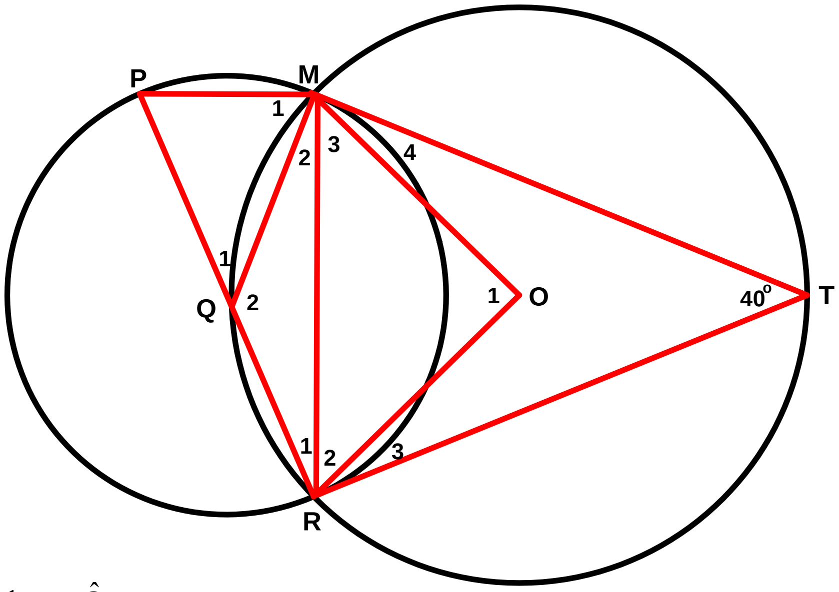
   Calculate *x*.



CONDITIONS FOR QUADRILATERAL TO BE CYCLIC

	If **OR** Then PQRS is a cyclic quad.	opp. int. angles suppl.
	If **OR** Then PQRS is a cyclic quad.	Angles in the same seg.
	If Then PQRS is a cyclic quad.	ext. angle equal to int. opp. angle

3. The diagram shows circles with centres Q and O, and $M\hat{T}R = 40^0$.
MT and RT are **not** necessarily tangents to the smaller circle.



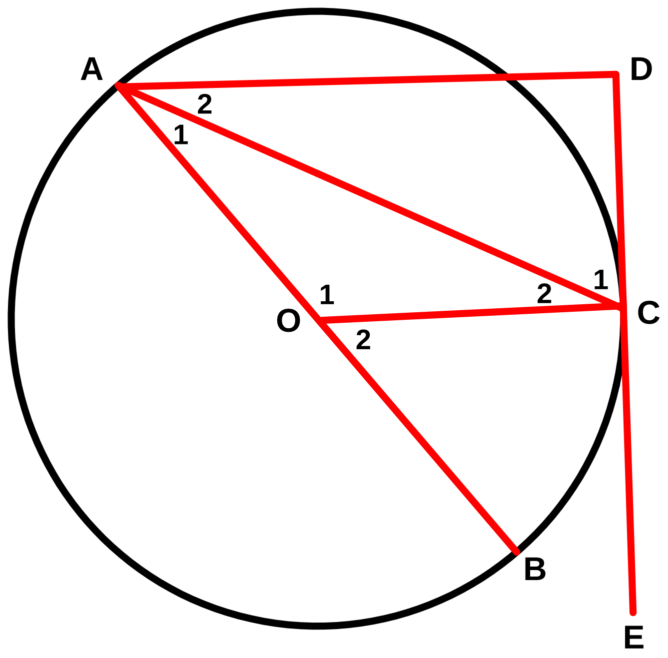
Determine: 3.1 $\hat{Q}_2$

3.2 $\hat{O}_1$

3.3 $P\hat{M}O$

3.4 $\hat{P}$

4. In the accompanying figure, AB is a diameter of the circle with centre O. DC is a tangent to the circle at point C. Chord AC is drawn. D is a point on the tangent DC so that $\hat{A}_1 = \hat{A}_2$.

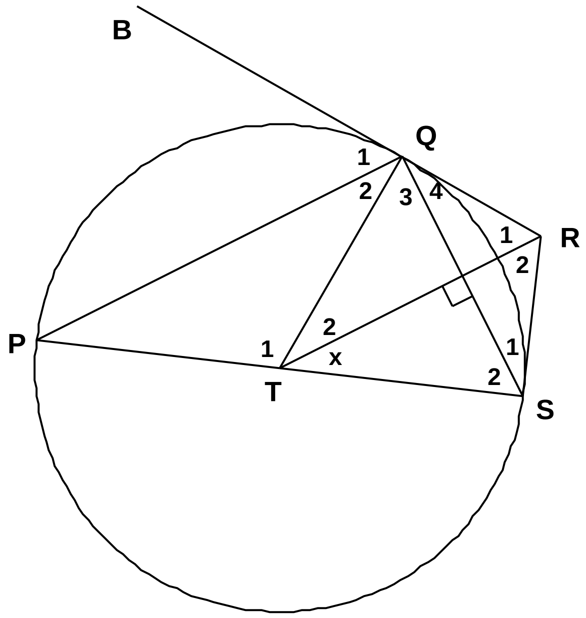


Prove that:

4.1 AD ∥ OC

4.2 $A\hat{D}C = 90°$

5. In the figure PS is a diameter of the circle with centre T. BQ is a tangent to the circle and TR is perpendicular to QS. $R\hat{T}S = x$.

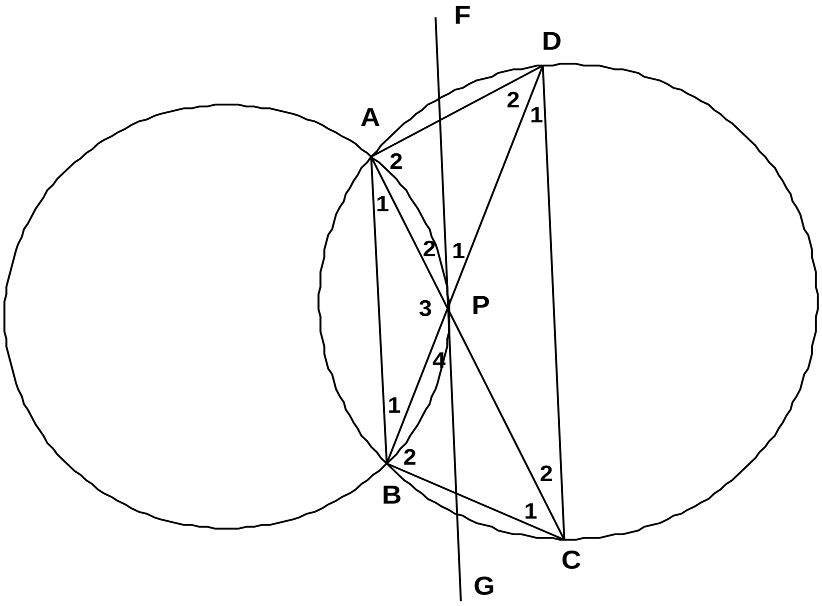


5.1 Prove that TR ∥ PQ.

5.2 Determine, with reasons, other four angles each equal to x.

5.3 Prove that TQRS is a cyclic quadrilateral.

6. In the figure below, diagonals AC and BD of cyclic quadrilateral ABCD intersect at P such that AP = PB. FPG is a tangent to circle



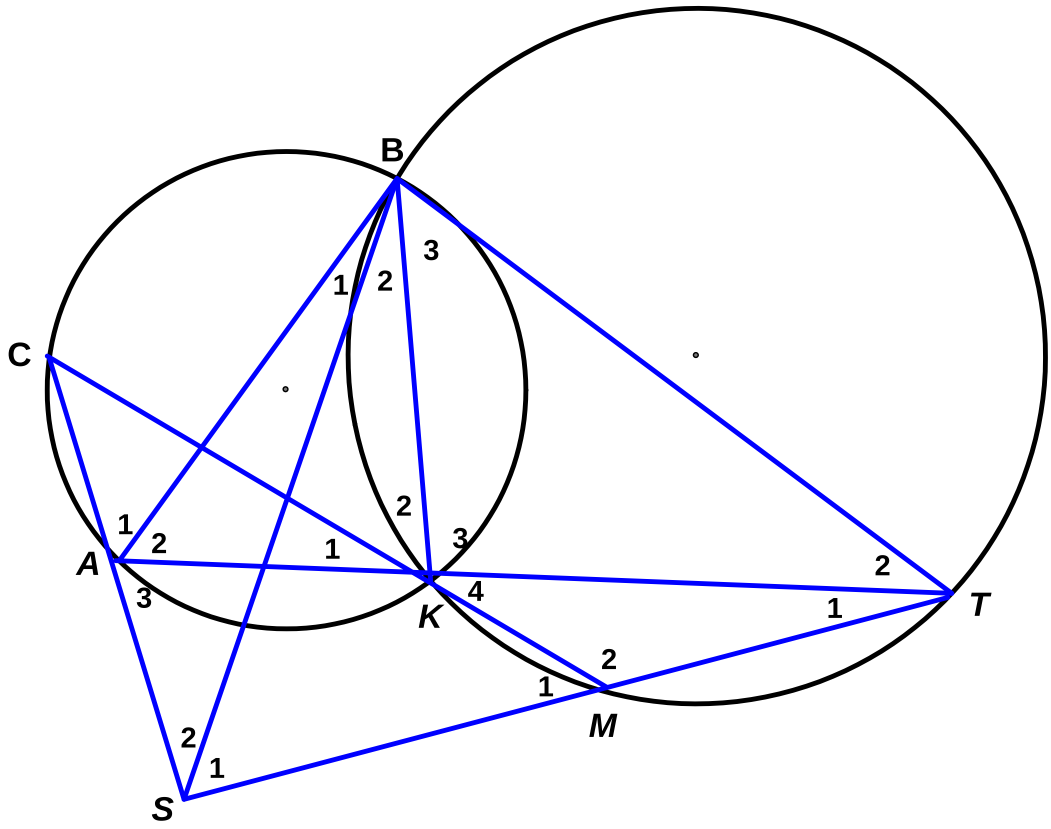
ABP.

Prove that:

6.1 FG ∥ DC

6.2

7. The sketch below shows circles BKAC and KMTB intersecting at K and B, and $A\hat{B}T = 90°$. AB and BT are not diameters, BT is not a tangent to the smaller circle, and AB is not a tangent to the larger circle.



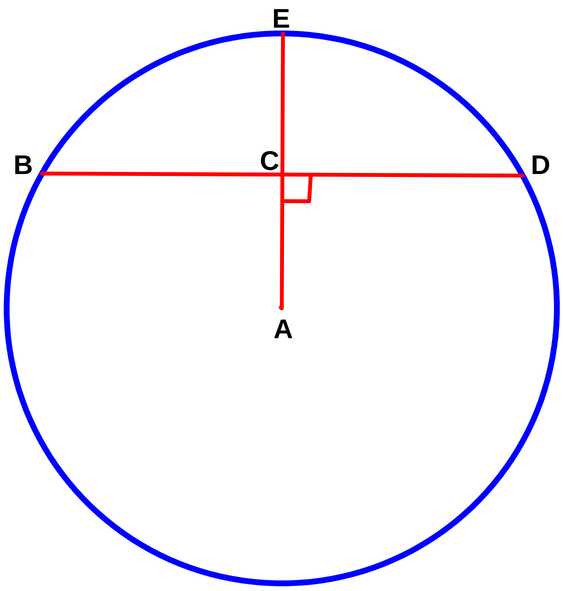
7.1 Prove that SABT is a cyclic quadrilateral.

7.2 Express in terms of .

7.3 Prove that

SOLUTIONS

GEOMETRY 1



1. BC = CD = 7cm ….. AC ⊥ BD
   Let AC = x

   Then the radius $r = x + 3$

   $AD^2 = AC^2 + CD^2$ … Pythag

   Thus $(x + 3)^2 = x^2 + 7^2$

   $\therefore x^2 + 6x + 9 = x^2 + 49$

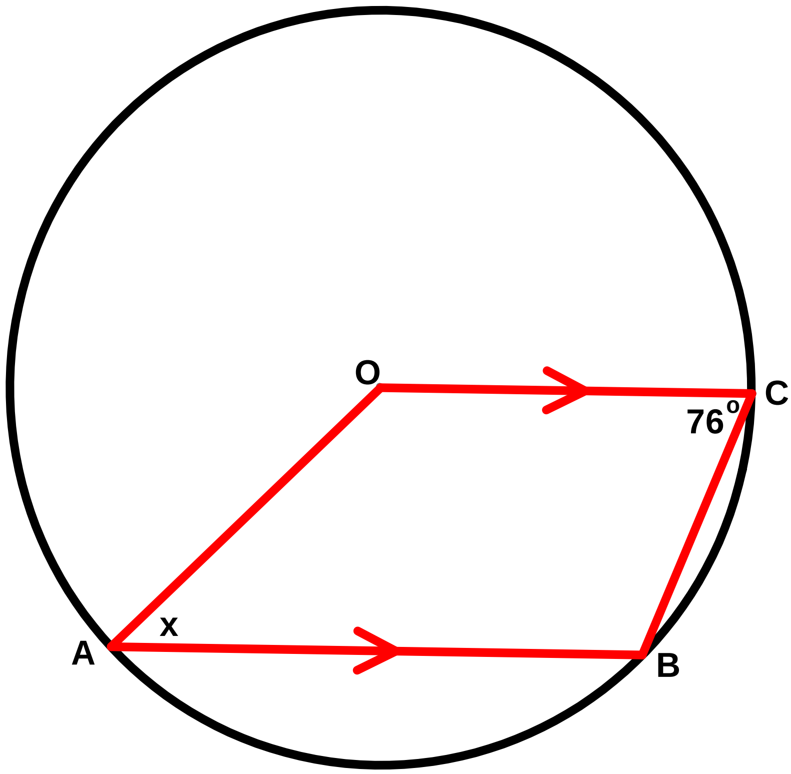
   $\therefore 6x = 40$

   $\therefore x = 6\frac{2}{3}$

   $\therefore r = 9\frac{2}{3}$

   $\therefore$ Area $= \pi\,(9\frac{2}{3})^2$

   $= 293,56$ cm²



2. $A\hat{B}C = 104°$ …. Co-int ∠'s;

   OC ∥ AB.

   $\therefore$ Reflex $A\hat{O}C = 2B$

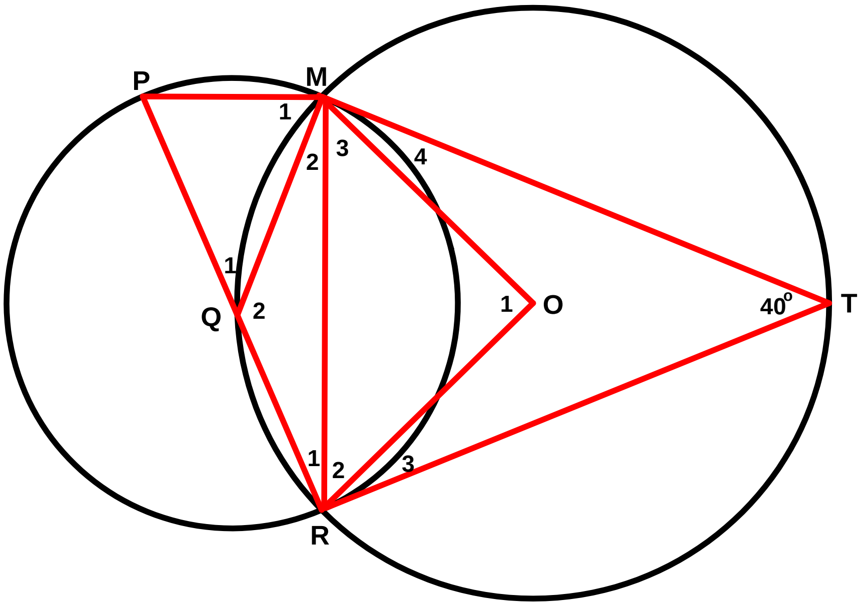
   $= 208°$

   $\therefore$ Obtuse $A\hat{O}C = 152°$

   $\therefore x = 28°$ … co-int ∠'s;

   OC ∥ AB.

3.



3.1 $\hat{Q}_2 = 140^\circ$ ….. opposite $\hat{T}$ in cyclic quad MQRT

3.2 $\hat{O}_1 = 80^\circ$ ….. $2 \times \hat{T}$ on the circumference

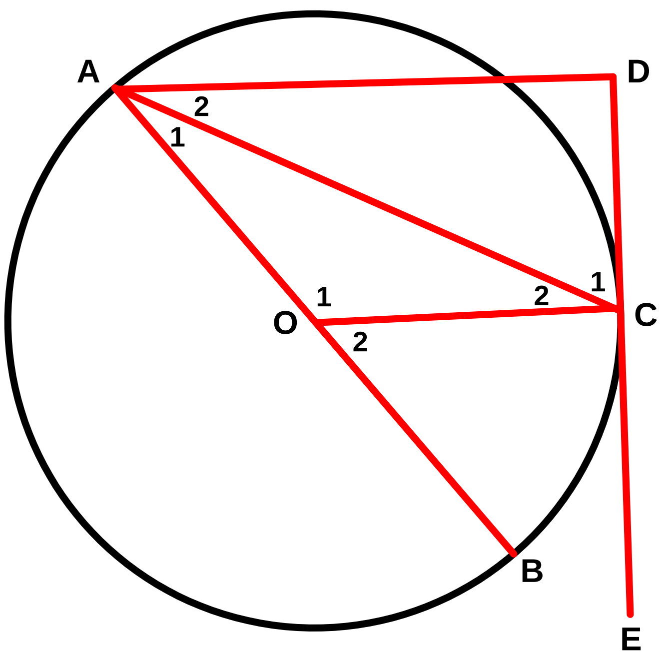
3.3 $P\hat{M}R = 90^\circ$ ….. ∠ in a semi-circle

$M_3 = 50^\circ$ ….. ∠ sum of isosceles Δ OMR

$\therefore P\hat{M}O = 140^0$

3.4 $\hat{P} = 70^\circ$ ….. $\hat{Q}_2$ is the exterior ∠ of isosceles Δ QMP

4.



4.1 $C_2 = A_1$ ….. OA = OC (radii)

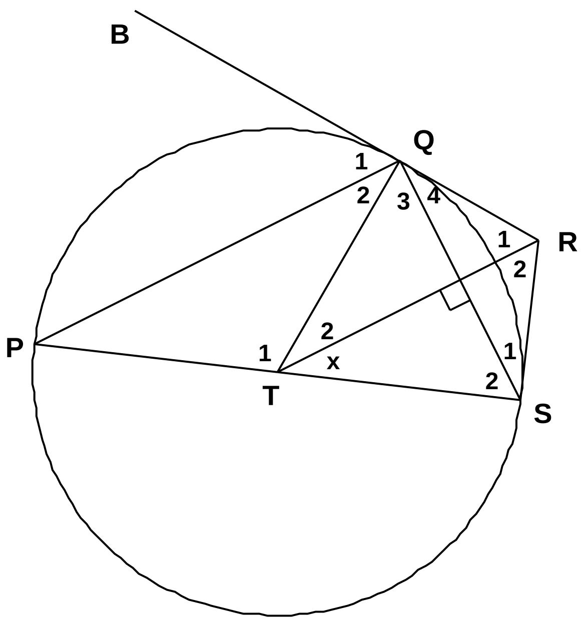
∴ $C_2 = A_2$ ….. $A_1 = A_2$ (given)

∴ AD ∥ OC ….. alternate angles equal

4.2 $O\hat{C}E = 90^\circ$ ….. tangent CE ⊥ radius OC

∴ $A\hat{D}C = 90^\circ$ ….. corresponding angles; AD ∥ OC

5.



5.1 $Q_{2+3} = 90^\circ$ …. ∠ in semi-circle

∴ TR ∥ PQ …. corresponding ∠'s equal

5.2 $T_2 = x$ …. TR is a line of symmetry of isos Δ TQS

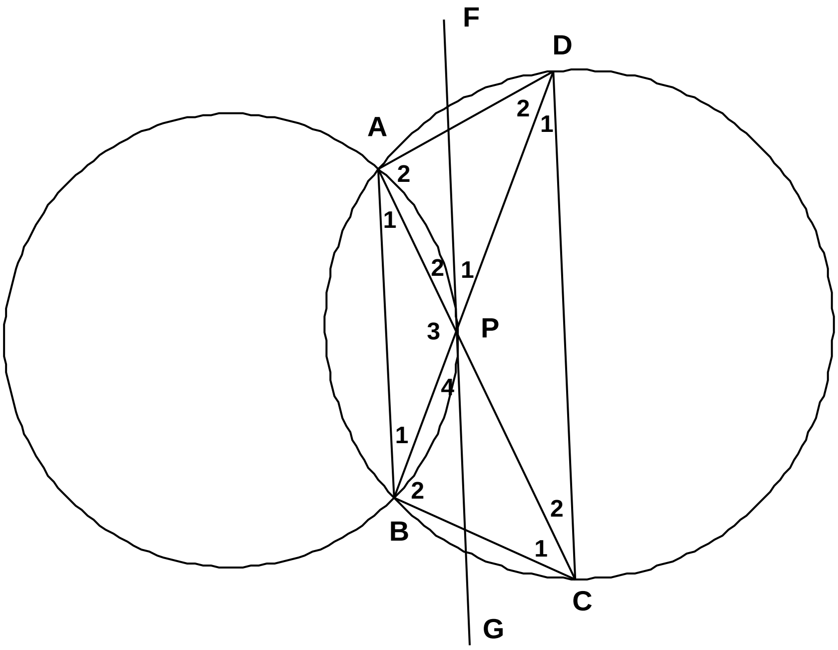
∴ P = x …. ½ T at the centre

∴ $Q_2 = x$ …. PT = TQ (radii)

$Q_4 = P = x$ …. tan BQR ; chord QS

5.3 $Q_4 = R\hat{T}S = x$. Thus TQRS is cyclic …. chord RS

6.



6.1) $\angle P_4 = \angle A_1$ ..... tang PG; chord PB

But $P_4 = P_1$ ..... vert opp angles and $A_1 = D_1$ ... chord BC

$\therefore\ P_1 = D_1$

$\therefore$ FG ∥ DC … alt∠'s equal

6.2 $P_4 = A_1$ ..... tang PG; chord PB

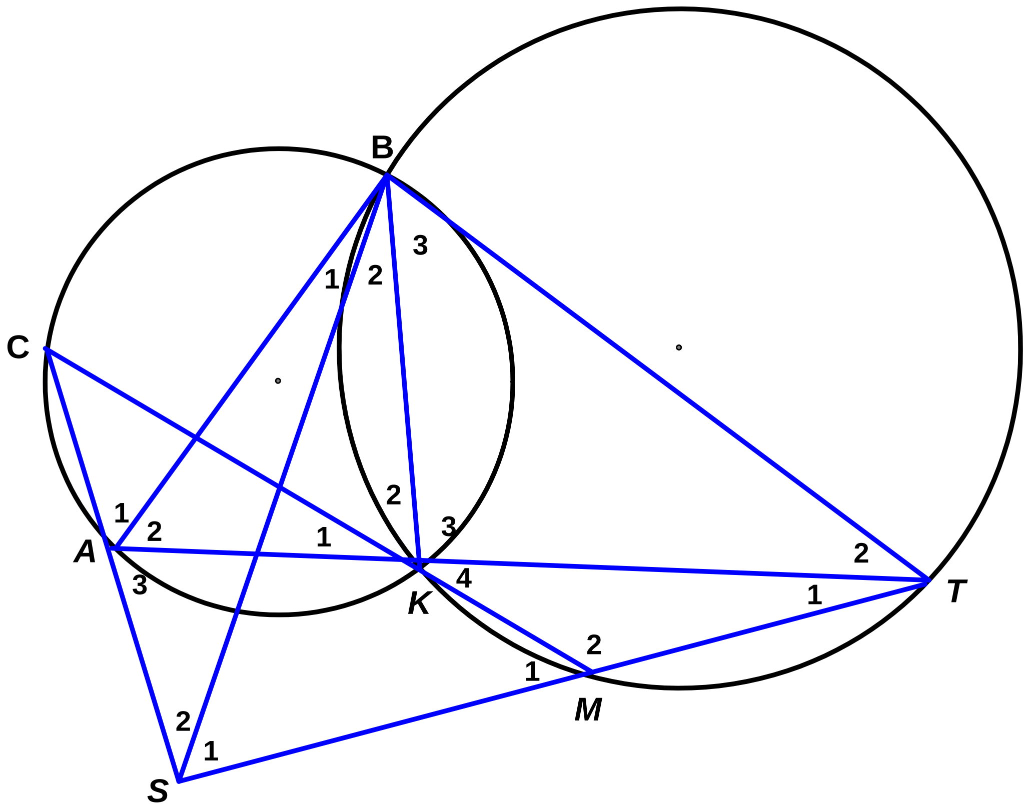
$P_2 = B_1$ …. tang FP; chord AP

But $A_1 = B_1$ …. equal chords AP and BP

$\therefore\ P_4 = P_2 = P_1$

$\therefore\ FP$ bisects $A\hat{P}D$

7.



7.1 $\hat{M}_1 = \hat{B}_3$ ….. exterior ∠ of cyclic quad BKMT

Let $M_1 = B_3 = x$

$\therefore$ $A\hat{B}K = 90° - x = \hat{C}$ ….. ∠'s in the same segment

$\therefore$ $A\hat{S}M = 90°$ ….. ∠ sum Δ CSM

$\therefore$ BAST is cyclic ….. $A\hat{B}T + A\hat{S}M = 180°$

7.2 Let $T_2 = y$

$\therefore$ $S_2 = y$ ….. same chord AB

$\therefore$ $S_1 = 90° - S_2 = 90° - T_2$

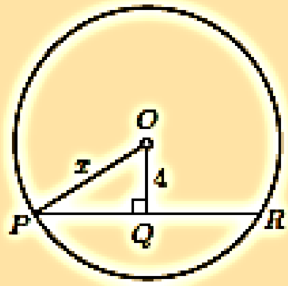
7.3 $M_1 = K_4 + T_1$ ….. ext ∠ of Δ

But $M_1 = B_3$, $K_4 = K_1$ (vert opp ∠'s) and $T_1 = B_1$ (chord AS)

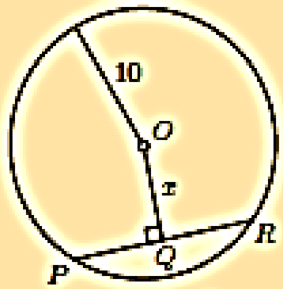
$\therefore\ \hat{B}_3 = \hat{K}_1 + \hat{B}_1$

MORE ACTIVITIES

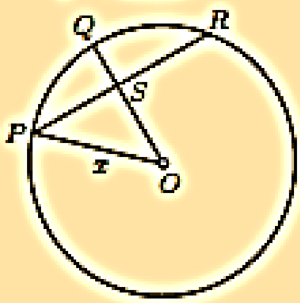
1. In the circle with centre O, $OQ \perp PR$, $OQ = 4$ units and $PR = 10$. Determine x.



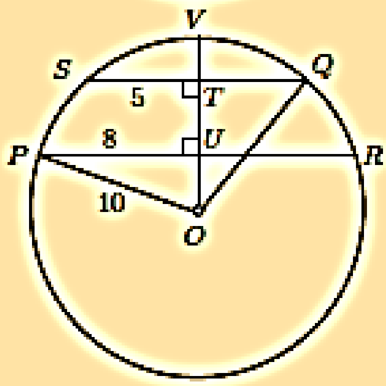
2. In the circle with centre O and radius $= 10$ units, $OQ \perp PR$ and $PR = 8$. Determine x.



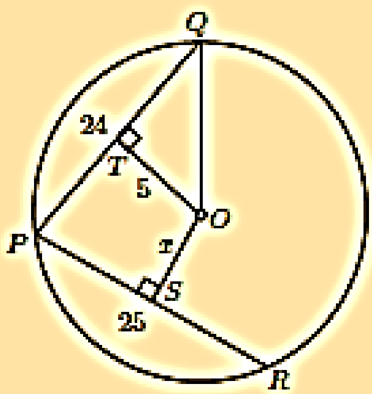
3. In the circle with centre O, $OQ \perp PR$, $PR = 12$ units and $SQ = 2$ units. Determine x.



4. In the circle with centre O, $OT \perp SQ$, $OT \perp PR$, $OP = 10$ units, $ST = 5$ units and $PU = 8$ units. Determine TU.

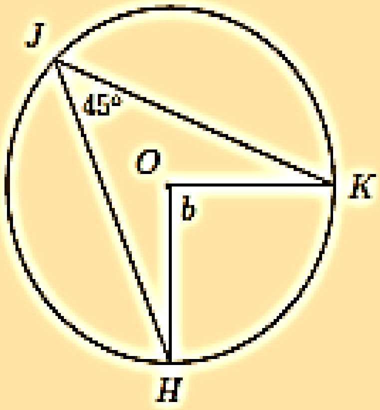


5. In the circle with centre O, $OT \perp QP$, $OS \perp PR$, $OT = 5$ units, $PQ = 24$ units and $PR = 25$ units. Determine $OS = x$.

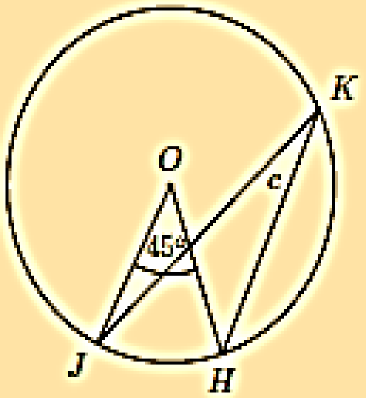


Given O is the centre of the circle, determine the unknown angle in each of the following diagrams:

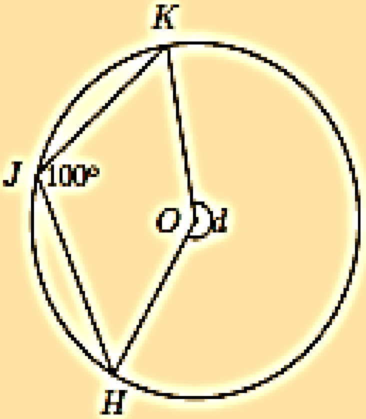
1.



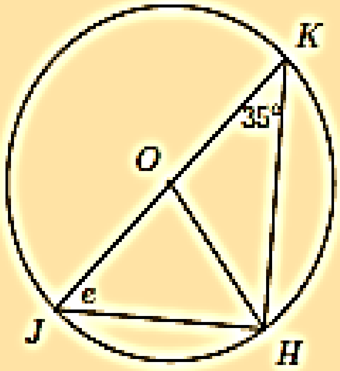
2.



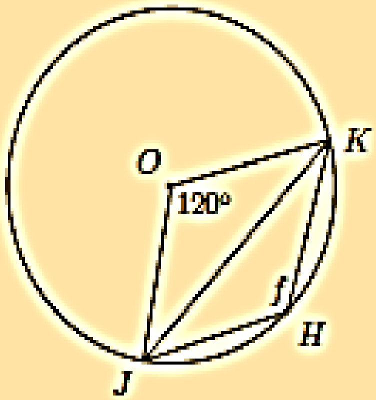
3.

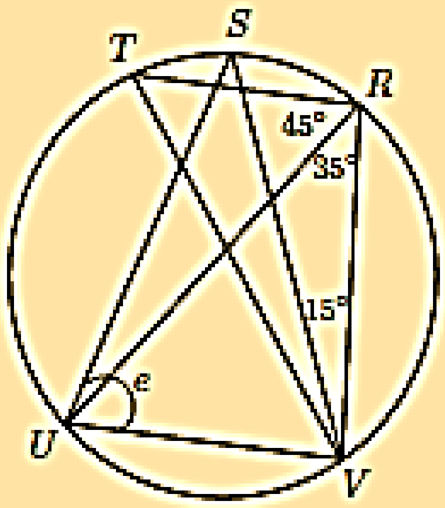


4.



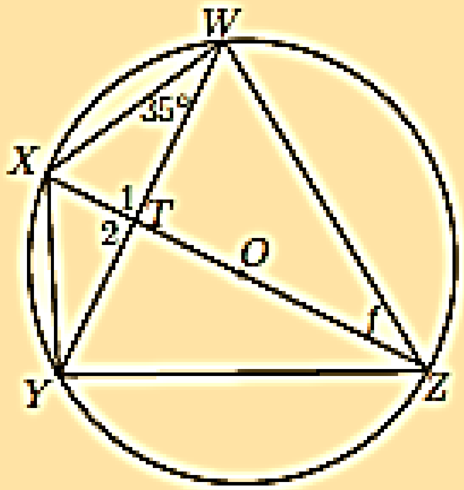
5.





a) Given $T\hat{V}S = S\hat{V}R$, determine the value of e.

b) Is TV a diameter of the circle? Explain your answer.



Given circle with centre O, $WT = TY$ and $X\hat{W}T = 35°$. Determine f.