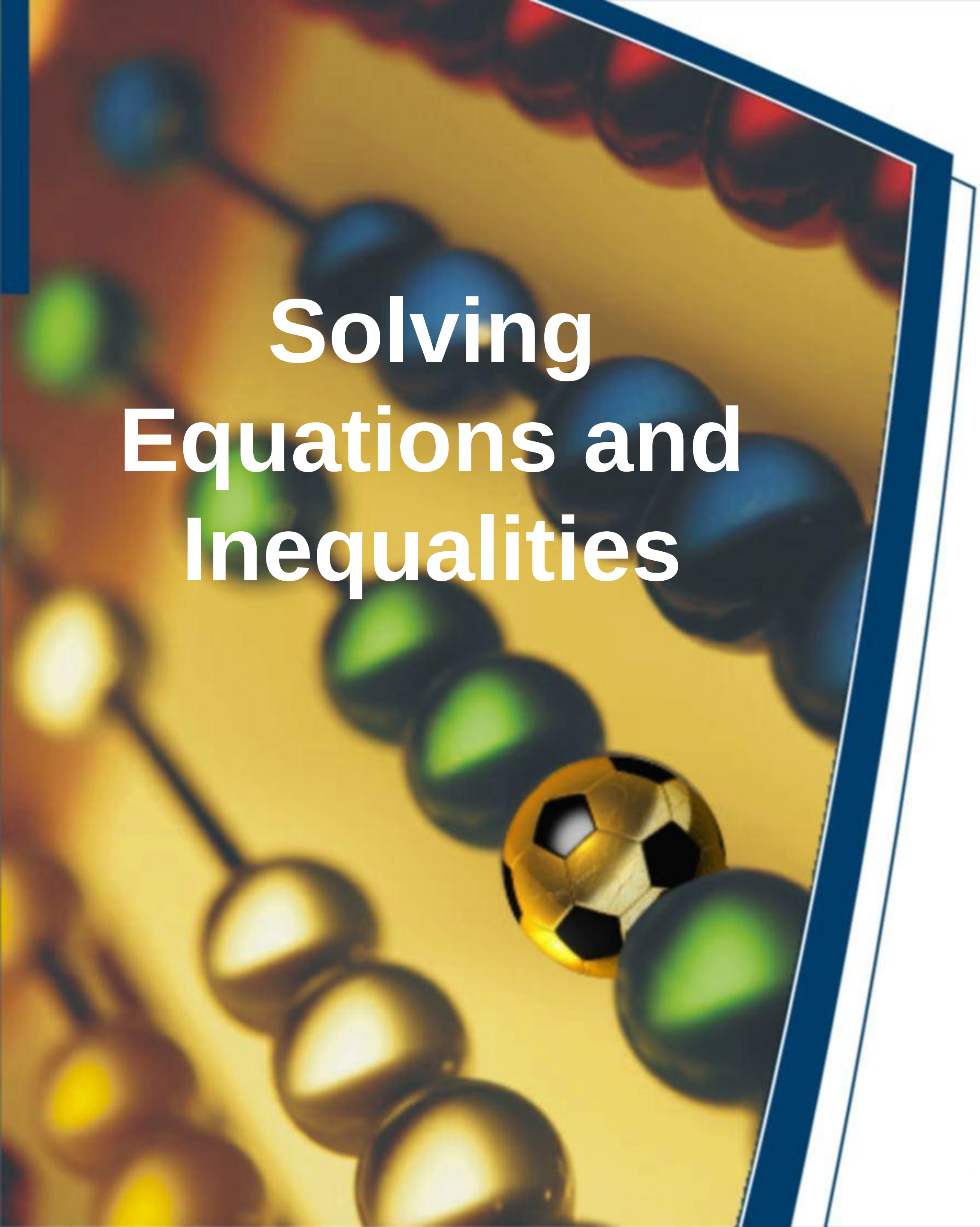
Solving Equations and Inequalities



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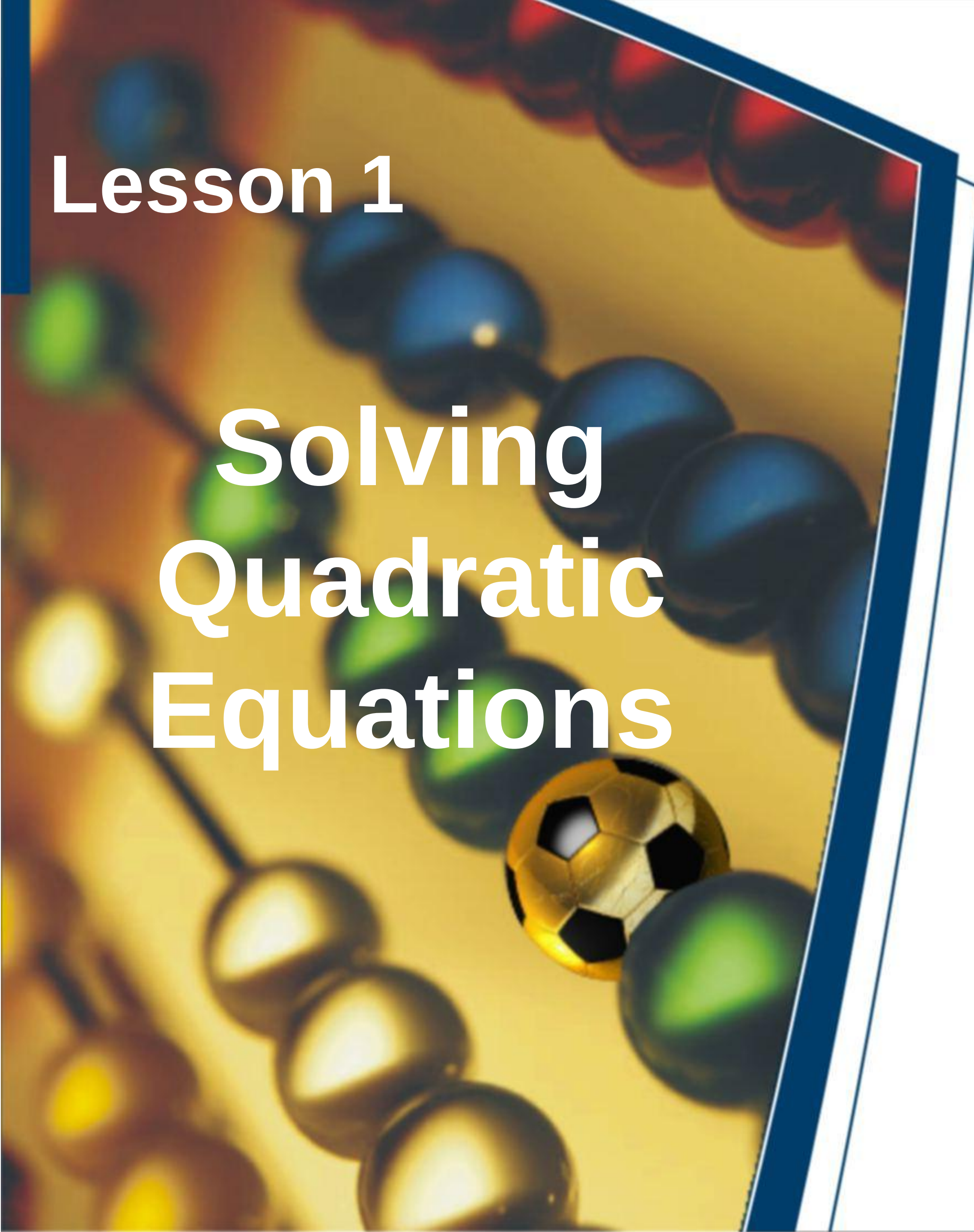
Outcomes for this DVD

In this DVD you will:

- Solve quadratic equations — LESSON1.
- Solve quadratic inequalities — LESSON 2.
- Solve general inequalities — LESSON 3.
- Solve systems of equations — LESSON 4.



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Lesson 1

Solving Quadratic Equations

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What is a Quadratic Equation?

Definition of quadratic equation:

- A quadratic equation is an equation that can be written as $ax^2 + bx + c = 0$ where a, b and $c \in \mathbb{R}$ and $a \neq 0$.

- $ax^2 + bx + c = 0$ is also known as the standard form of the quadratic equation.

Strategy for solving Quadratic Equations

Write the quadratic equation in standard form:

The roots are solution's of this equation

$$ax^2 + bx + c = 0; a \neq 0$$

Either $x - a = 0$ or $x - \beta = 0$

Factorize if possible:

$$a(x - \alpha)(x - \beta) = 0$$

Roots

or

Use Formula:

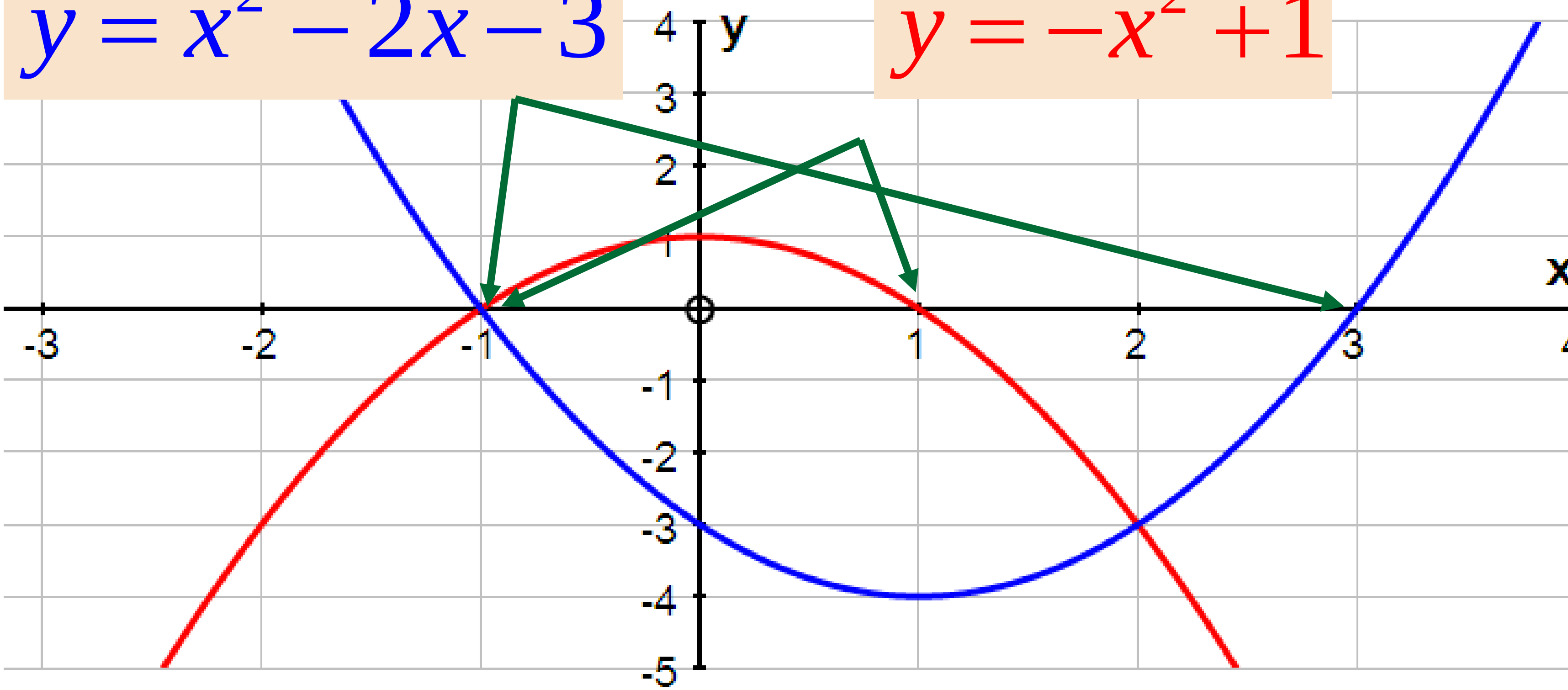
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Roots – The Graphical Interpretation

i.e. x - intercepts for the graph of $y = ax^2 + bx + c$

$y = x^2 - 2x - 3$

$y = -x^2 + 1$



Example 1: Solving a Quadratic Equation

1. $(x-2)(x+1)=0$

This is already given in factorised form:

$$\text{Thus; } (x-2)(x+1)=0$$

$$\Leftrightarrow x-2=0 \text{ or } x+1=0$$

$$\Leftrightarrow x=2 \text{ or } x=-1$$

Example 2: Solving a Quadratic Equation

2. Solve: $3x^2 = 27$

$\Leftrightarrow 3x^2 - 27 = 0$ (Standard form)

$\Leftrightarrow x^2 - 9 = 0$ (Divide by 3)

$\Leftrightarrow (x+3)(x-3) = 0$ (Factorise)

$\Leftrightarrow x = -3$ or 3 (Solutions- Check!)

Example 3: Solving a Quadratic Equation

3. Solve: $2x(x-3)=20$

$$\Leftrightarrow 2x^2 - 6x - 20 = 0$$

$$\Leftrightarrow x^2 - 3x - 10 = 0$$

$$\Leftrightarrow (x-5)(x+2) = 0$$

$$\Leftrightarrow x = 5 \text{ or } x = -2$$

Tutorial 1: Solving Quadratic Equations

Solve the following quadratic equations:

1. $12 - x = x^2$

2. $2 = 15k^2 - k$

3. $3 + 14x = 5x^2$

PAUSE DVD

- Do Tutorial 1
- Then View Solutions

Tutorial 1 Example 1: Suggested Solution

1. Solve: $12 - x = x^2$

$$\Leftrightarrow x^2 + x - 12 = 0$$

$$\Leftrightarrow (x+4)(x-3) = 0$$

$$\Leftrightarrow x = -4 \text{ or } x = 3$$

Tutorial 1 Example 2: Suggested Solution

2. Solve: $2 = 15k^2 - k$

$$\Leftrightarrow 15k^2 - k - 2 = 0$$

$$\Leftrightarrow (5k-2)(3k+1) = 0$$

$$\Leftrightarrow k = \frac{2}{5} \text{ or } k = -\frac{1}{3}$$

Tutorial 1 Example 3: Suggested Solution

3. Solve: $3+14x=5x^2$

$$\Leftrightarrow 5x^2-14x-3=0$$

$$\Leftrightarrow (5x+1)(x-3)=0$$

$$\Leftrightarrow x=-\frac{1}{5} \text{ or } x=3$$

Solving a Quadratic Equation by Completing the Square

When a quadratic equation does not factorise,

we can solve the equation by completing the square.

Especially when:

$\Delta = b^2 - 4ac$ is positive but not a perfect square.

Note: Completing the square means that we add and subtract a number to an existing quadratic expression so that part of the expression becomes a perfect square.

Example 1: Creating a Perfect Square

(1) What must be added to $x^2 - 6x$,

to make this expression a perfect square?

We rewrite $x^2 - 6x$ as:

- Assume that leading coefficient is 1

$$x^2 - 6x = x^2 - 6x + \left(\frac{-6}{2}\right)^2 - 9$$

$$= \left(x^2 - 6x + (-3)^2\right) - 9 = (x-3)^2 - 9$$

- Number to be added and subtracted can be calculated as follows:

$$\left(\frac{1}{2} \times \text{ Coefficient of Middle Term}\right)^2$$

Example 2: Convert the Quadratic Expression into a Perfect Square

(2) To make $x^2 + 4x$ a perfect square

we add and subtract $\left(\frac{1}{2} \times 4\right)^2 = 2^2 = 4$ to it.

$x^2 + 4x$ can be rewritten as:

$$x^2 + 4x = \left(x^2 + 4x + 2^2\right) - 4 = (x+2)^2 - 4$$

Solving Quadratic Equations by Completing the Square

Instruction:	Solve: $2x^2-4x-7=0$
Take constant to RHS	$2x^2-4x=7$
Divide both sides by coeff. of x^2	$x^2-2x=\frac{7}{2}$
Add $\left[\frac{1}{2}\text{coeff. of } x\right]^2$ to both sides	$x^2-2x+1=\frac{7}{2}+1$
The 3 terms on LHS form a perfect square	$(x-1)^2=1+3,5$ $\therefore\ (x-1)^2=4,5$
Take square roots on both sides:	$x-1=\pm\sqrt{4,5}$
Write final solution:	$x=1\pm\sqrt{4,5}$

Solving $ax^2 + bx + c = 0$ by completing the square

Solve $ax^2 + bx + c = 0$

by completing the square.

$$\Leftrightarrow ax^2 + bx = -c \Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Leftrightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Leftrightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \Leftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$$

$$\Leftrightarrow x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

This is the formula to solve any quadratic equation.

Tutorial 2: Solve quadratic equations by completing the square

Solve for x by completing the square. (Write answer(s) correct to 2 decimal digits where necessary.)

1. $x^2 - 4x + 2 = 0$
2. $3x^2 - 2x - 2 = 0$

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- Do Tutorial 2
- Then View Solutions

Tutorial 2 Problem 1: Suggested Solution

1. $x^2 - 4x + 2 = 0$

$$\Leftrightarrow x^2 - 4x = -2$$

$$\Leftrightarrow x^2 - 4x + (-2)^2 = (-2)^2 - 2$$

$$\Leftrightarrow (x-2)^2 = 2$$

$$\Leftrightarrow x - 2 = \pm\sqrt{2}$$

$$\Leftrightarrow x \approx 2 \pm 1,41$$

$$\Leftrightarrow x \approx 3,41 \text{ or } 0,59$$

Tutorial 2 Problem 2: Suggested Solution

2. $3x^2 - 2x - 2 = 0 \Leftrightarrow 3x^2 - 2x = 2$

$$\Leftrightarrow x^2 - \frac{2}{3}x = \frac{2}{3}$$

$$\Leftrightarrow x^2 - \frac{2}{3}x + \left(-\frac{1}{3}\right)^2 = \left(-\frac{1}{3}\right)^2 + \frac{2}{3}$$

$$\Leftrightarrow \left(x - \frac{1}{3}\right)^2 = \frac{1}{9} + \frac{2}{3} \Leftrightarrow \left(x - \frac{1}{3}\right)^2 = \frac{1+6}{9}$$

$$\Leftrightarrow \left(x - \frac{1}{3}\right) = \frac{\pm\sqrt{7}}{3}$$

$$\Leftrightarrow x = \frac{1 \pm \sqrt{7}}{3} \approx \frac{1 \pm 2,65}{3} \Leftrightarrow x \approx 1,22 \text{ or } -0,55$$

Solving a Quadratic Equation by using the Formula

If $ax^2 + bx + c = 0$, then by completing the square we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

This is the general formula for solving quadratic equations.

Using the Formula to Solve Quadratic Equations

Solve for *x* by by using the general quadratic formula:

$$2x^2 - 7x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Know that:

$$a = 2$$

$$b = -7$$

$$c = 1$$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-7)^2 - 4(2)(1)$$

$$= 49 - 8 = 41$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\Rightarrow x = \frac{7 \pm \sqrt{41}}{4}$$

$$\Rightarrow x = \frac{7 + \sqrt{41}}{4} \text{ or } x = \frac{7 - \sqrt{41}}{4}$$

$$\Rightarrow x \approx 0{,}15 \text{ or } 3{,}35$$

Tutorial 3: Solving quadratic equations

Solve the following quadratic equations

using any suitable method:

1) $x(x-8)=-12$

2) $x^2+2x-5=0$

(Answers correct to 2 decimal places, where applicable)

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- Do Tutorial 3
- Then View Solutions

Tutorial 3 Problem 1: Suggested Solution

$$1.\ x(x-8)=-12$$

$$\Leftrightarrow x^2-8x+12=0$$

$$\Delta=(-8)^2-4(1)(12)=64-48=4^2$$

$\because \Delta$ is a perfect square

$x^2-8x+12$ can be factorised.

$$\therefore x^2-8x+12=0$$

$$\Leftrightarrow (x-2)(x-6)=0$$

$$\Leftrightarrow x=2 \text{ or } 6$$

Tutorial 3 Problem 2: Suggested Solution

2) $x^2 + 2x - 5 = 0$

$\Rightarrow \Delta = 2^2 - 4(1)(-5) = 24$

$\therefore \Delta$ is not a perfect square

$\Rightarrow$ Use formula

$$x^2 + 2x - 5 = 0$$

$$\Leftrightarrow x = \frac{-2 \pm \sqrt{24}}{2(1)}$$

$$\Leftrightarrow x \approx \frac{-2 \pm 4,899}{2}$$

$$\Leftrightarrow x \approx 1,45 \text{ or } -3,45$$

Lesson 2

Solving Quadratic Inequalities

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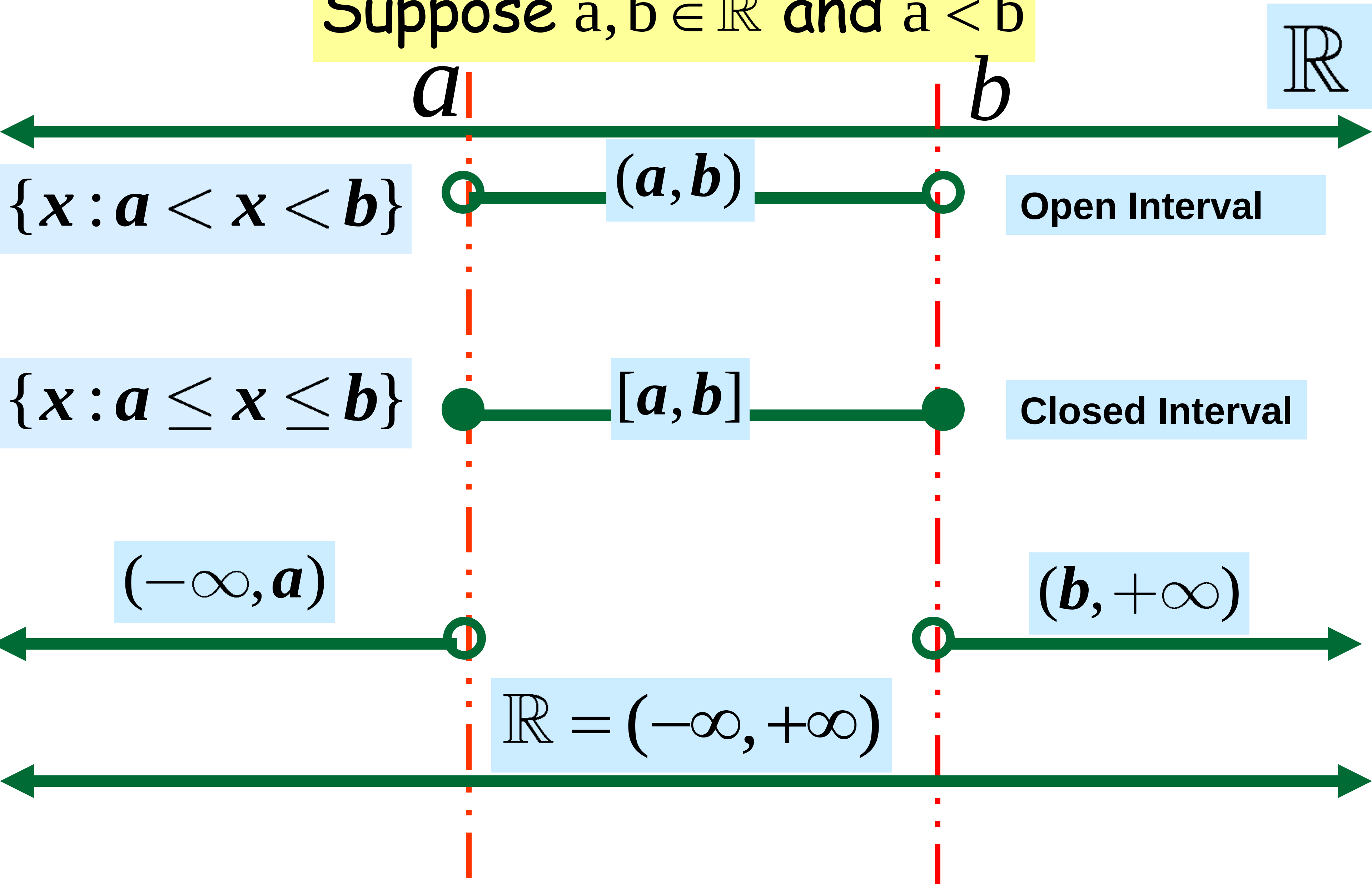
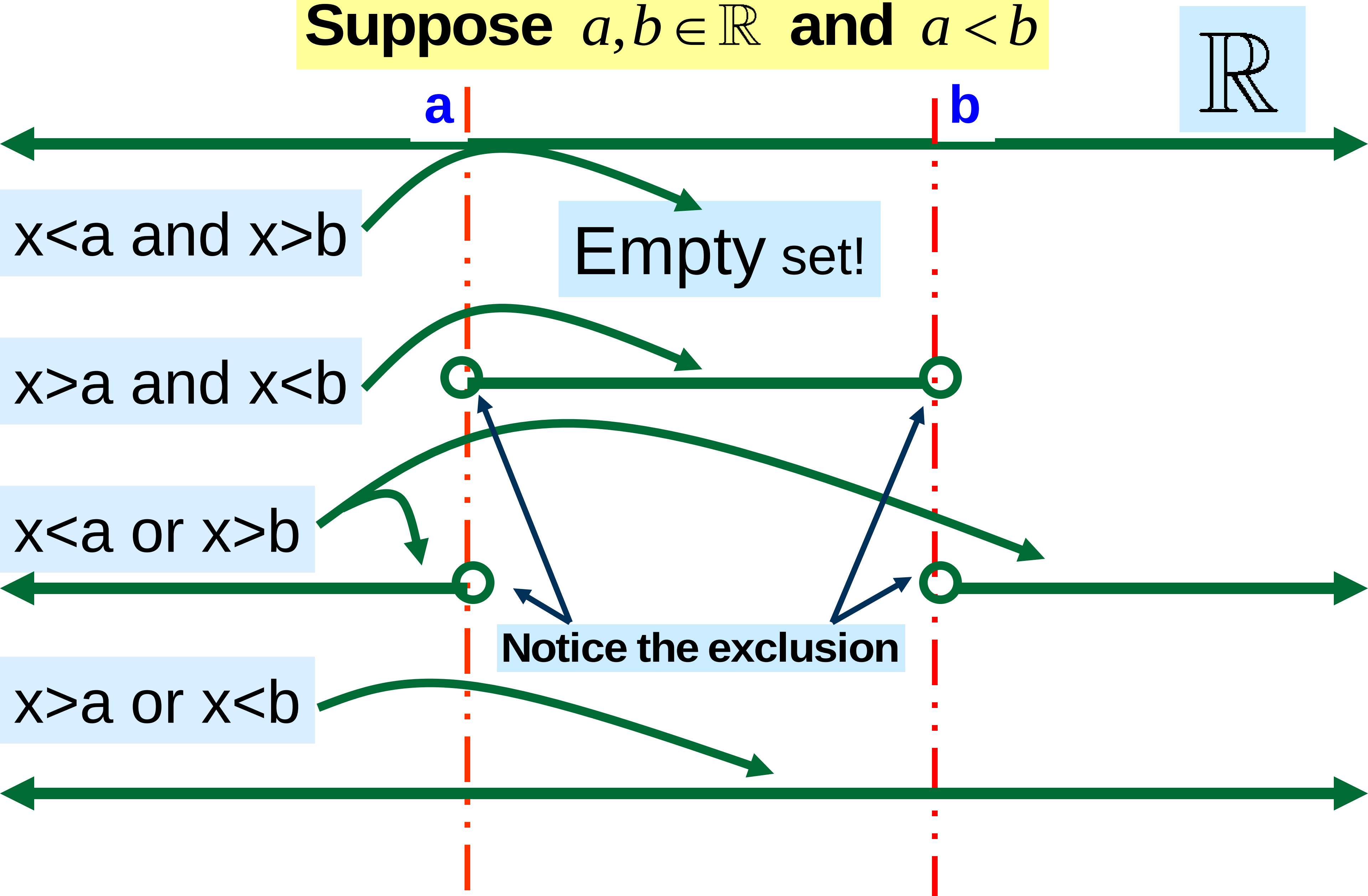
Sub-intervals of the Real Number Line

Suppose $a, b \in \mathbb{R}$ and $a < b$



The use of OR / AND

Suppose $a, b \in \mathbb{R}$ **and** $a < b$



Procedure for Graphical Solution of Non-linear Quadratic Inequalities

- Write all non-zero terms on left hand side of inequality.

  e.g. $ax^2 + bx + c \leq 0$ or $ax^2 + bx + c > 0$

- Sketch the parabola $y = ax^2 + bx + c$

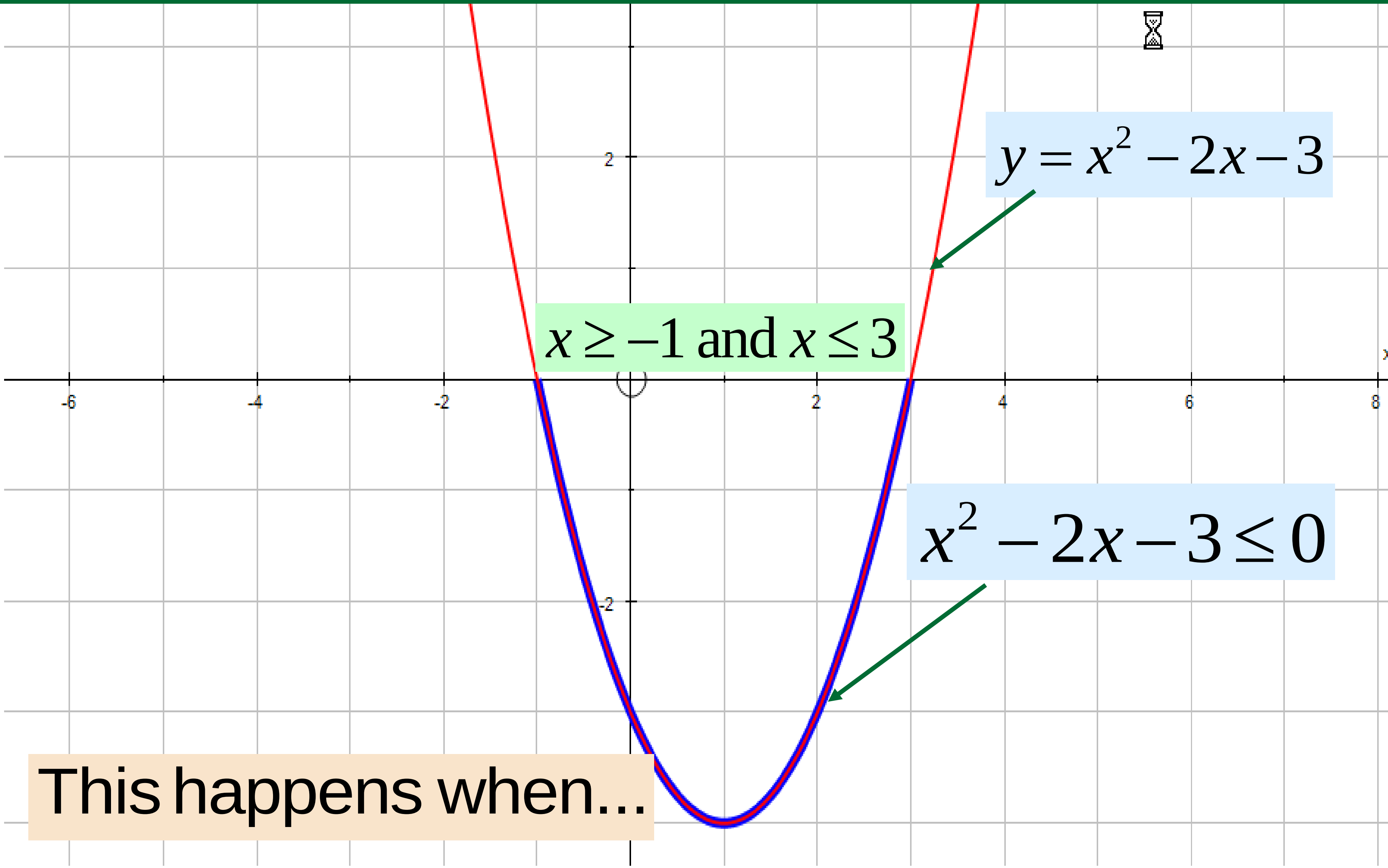
- $ax^2 + bx + c \leq 0$

  $\Rightarrow$ For which $x-$values does the parabola lie below or on the $x-$axis?

- $ax^2 + bx + c > 0$

  $\Rightarrow$ For which $x-$values does the parabola lie above the $x-$axis?

Graphical View of an Inequality



$y = x^2 - 2x - 3$

$x \geq -1$ and $x \leq 3$

$x^2 - 2x - 3 \leq 0$

This happens when...

Tutorial 4: Quadratic inequalities

Solve the following quadratic inequalities:

1. $3x^2 \geq 48$ (leave answer in set-builder notation)
2. $x^2 - 3x \leq 10$ (leave answer in interval notation)
3. $x - 1 < -2x^2$ (draw graphical solution)

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- Do Tutorial 4
- Then View Solutions

Tutorial 4 Problem 1: Suggested Solution

$$1.\ 3x^2 \geq 48$$

$$\Leftrightarrow 3x^2 - 48 \geq 0$$

$$\Leftrightarrow x^2 - 16 \geq 0$$

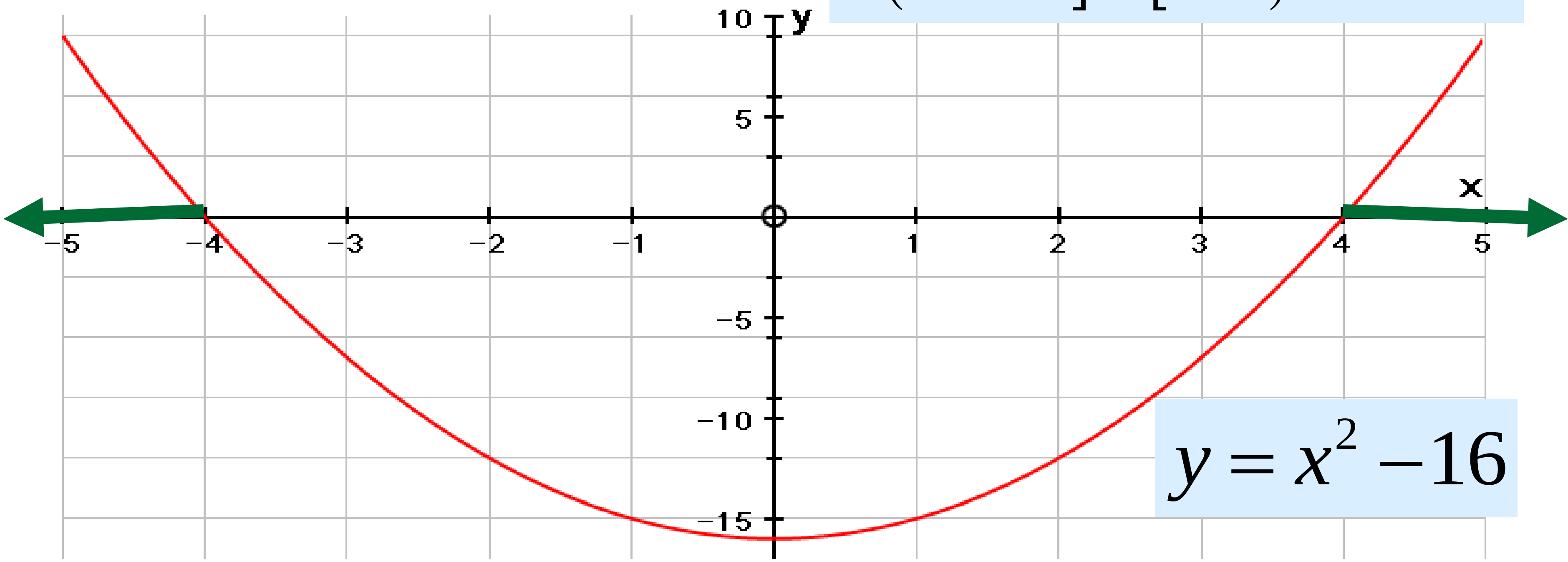
$$\Leftrightarrow (x+4)(x-4) \geq 0$$

$$\Leftrightarrow x \leq -4 \text{ or } x \geq 4$$

$\therefore$ Solution Set

$$= \{x : x \leq -4 \text{ or } x \geq 4;\ x \in \mathbb{R}\}$$

$$= (-\infty; -4] \cup [4; \infty)$$



$$y = x^2 - 16$$

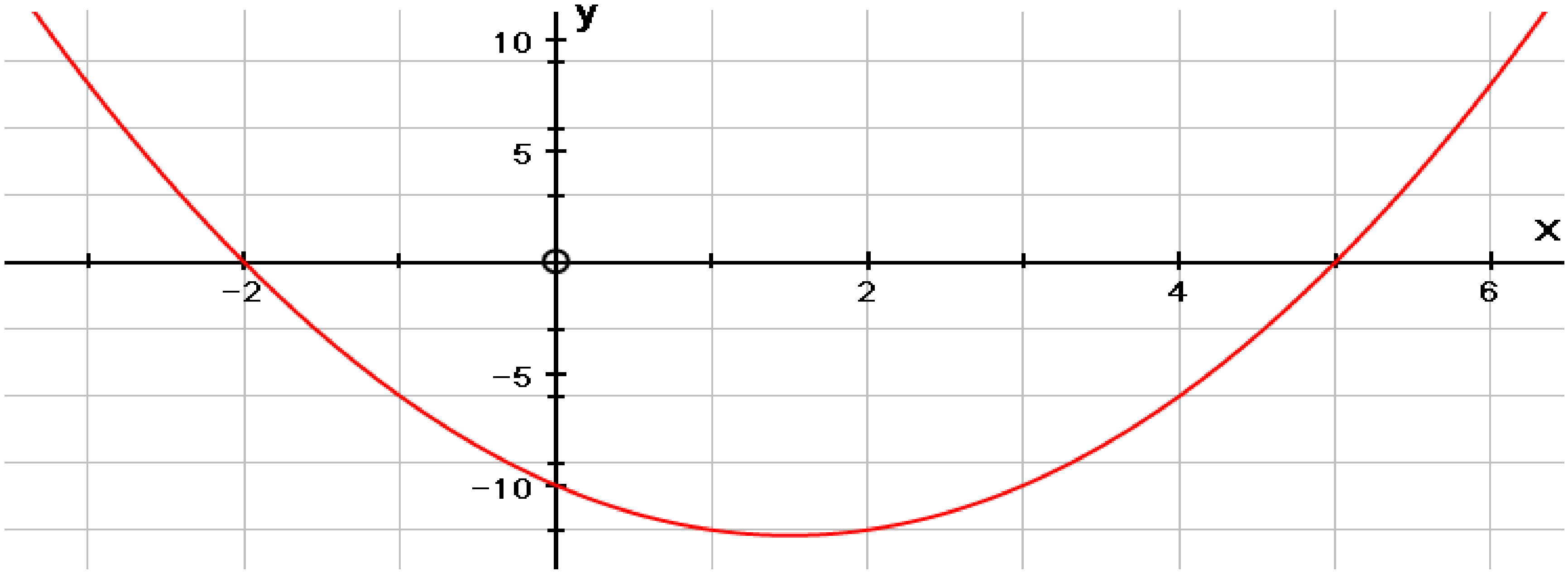
Tutorial 4 Problem 2: Suggested Solution

2. $x^2 - 3x - 10 \le 0$

$\Leftrightarrow (x+2)(x-5) \le 0$

$\Leftrightarrow -2 \le x \le 5$

Solution $x \in [-2; 5]$



Tutorial 4 Problem 3: Suggested Solution

3. Write in standard form and factorize:

$$x - 1 < -2x^2$$

$$\Leftrightarrow 2x^2 + x - 1 < 0$$

$$\Leftrightarrow (2x-1)(x+1) < 0$$

Sketch parabola defined by

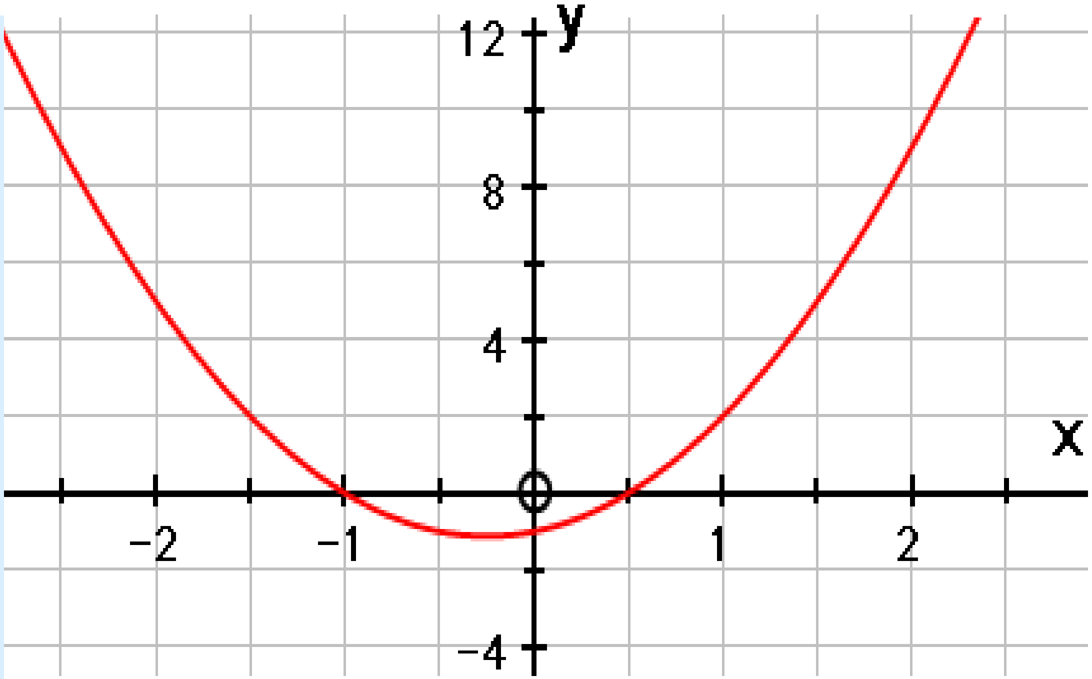
$$y = (2x-1)(x+1)$$

From sketch:

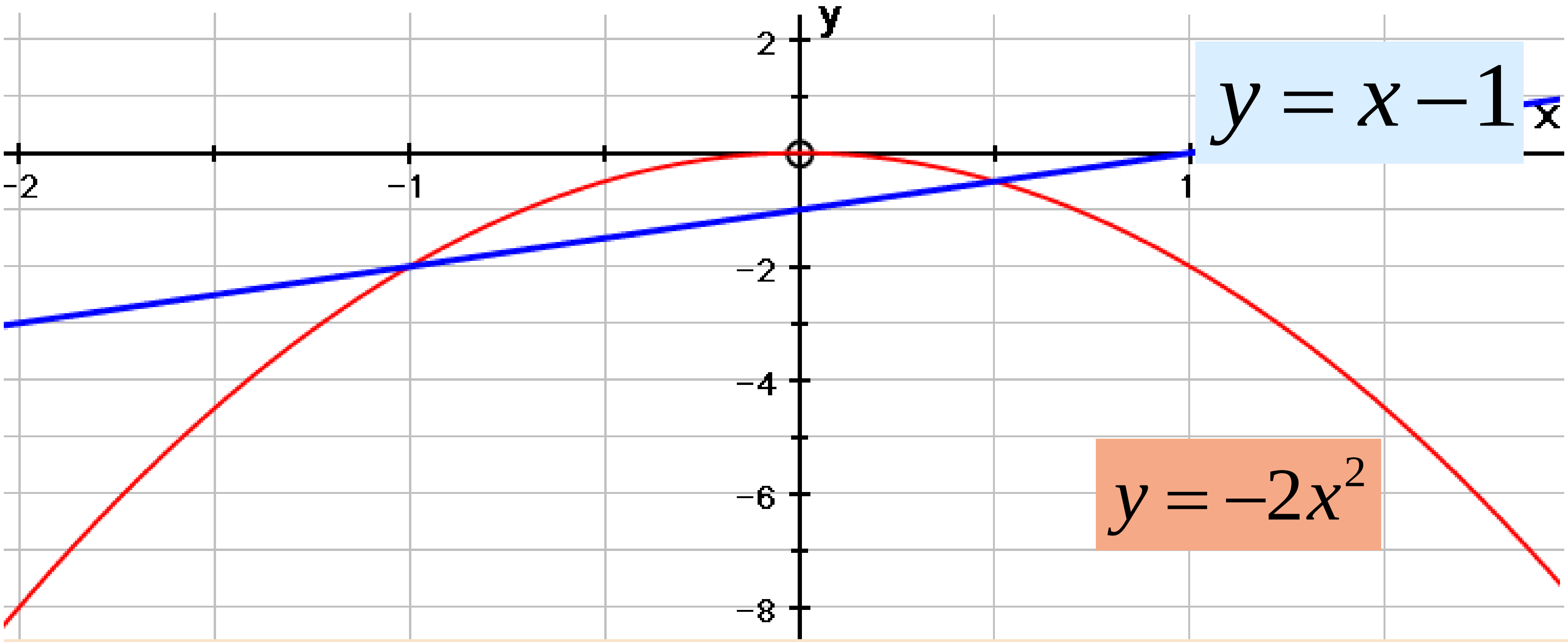
$$(2x-1)(x+1) < 0$$

$$\Rightarrow -1 < x \text{ and } x < \frac{1}{2}$$

$$\therefore x \in \left(-1; \frac{1}{2}\right)$$



Tutorial 4 Problem 3: Graphical Solution



We interpret the inequality $x - 1 < -2x^2$ graphically by finding the value/s of x for which the graph of $y = x - 1$ is below the graph of $y = -2x^2$. We see from the graph that these values lie between -1 and $\frac{1}{2}$

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Lesson 3

Solving General Inequalities

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Strategy to Solve General Inequalities

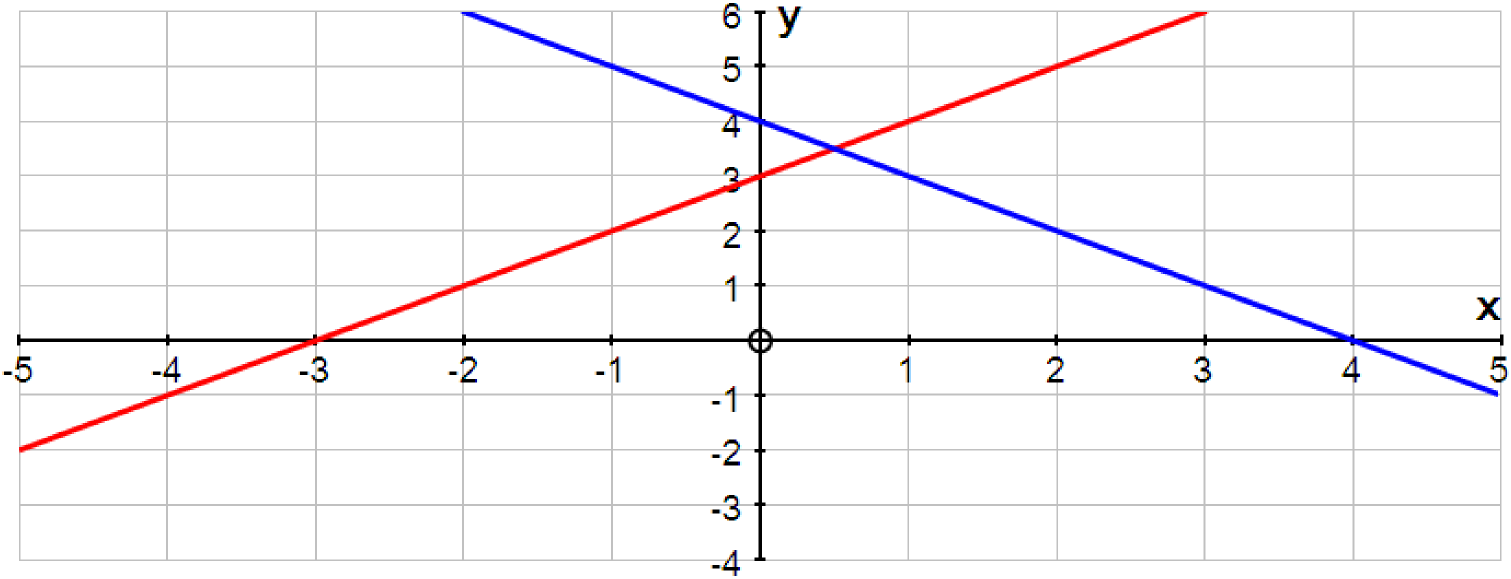
To solve inequalities like $\dfrac{x+2}{3-x}<0;\quad \dfrac{(x-3)(2-x)}{(4+x)}\geq 0;\quad \cdots$

we use the following method:

- Write all terms on LHS of inequality and 0 on RHS
- Write expression on LHS as a factorized fraction
- Find the value for which each factor in the expression is equal to zero
- Determine the sign of each linear factor between zero points, and hence the sign of the overall expression

Signs of Linear Factors

Consider the graphs of linear functions defined by $y = x + 3$ and $y = 4 - x$



Then $\begin{cases} x+3=0 & \text{when} & x=-3 \\ x+3>0 & \text{when} & x>-3 \\ x+3<0 & \text{when} & x<-3 \end{cases}$ and $\begin{cases} 4-x=0 & \text{when} & x=4 \\ 4-x>0 & \text{when} & x<4 \\ 4-x<0 & \text{when} & x>4 \end{cases}$

Example 1: Solve a Quadratic Inequality Algebraically

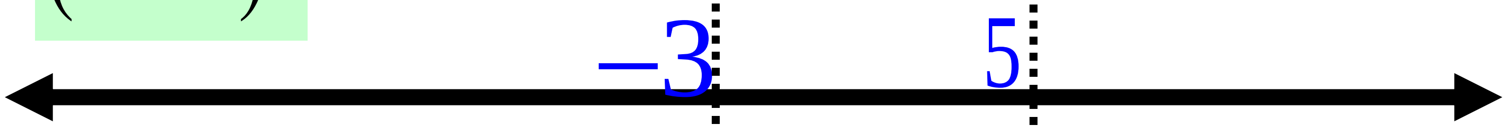
1. Determine the values of x such that

$$x^2 - 2x \le 15$$

Solution:

$$x^2 - 2x - 15 \le 0 \Rightarrow (x-5)(x+3) \le 0 \Longrightarrow x \in [-3; 5]$$

$(x-5)$:	$-$	$-$	$-$	0	$+$
$(x+3)$:	$-$	0	$+$	$+$	$+$
$(x-5)(x+3)$:	$+$	0	$-$	0	$+$

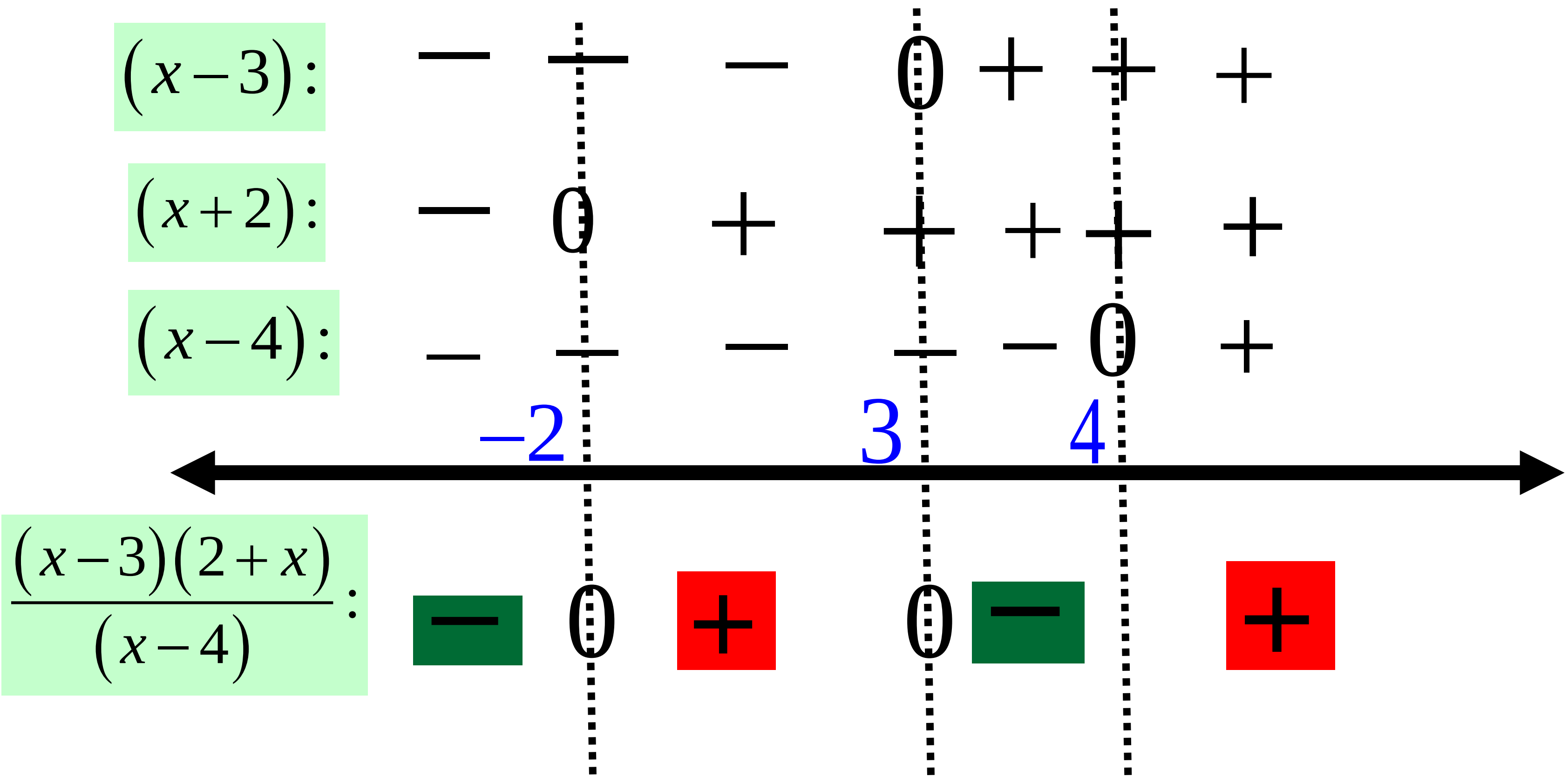


Example 2: Solve a Quadratic Inequality Algebraically

2. Determine the values of x such that

$$\frac{(x-3)(2+x)}{(x-4)} \geq 0$$

$$\Rightarrow x \in [-2;3] \cup (4;\infty)$$



Tutorial 5: Solve General Inequalities

Solve:

1. $2x^2 - 3x < 0$

2. $\dfrac{(x-3)}{(2+x)(x-4)} \le 0$

PAUSE DVD

- Do Tutorial 5
- Then View Solutions

Tutorial 5 Problem 1: Suggested Solution

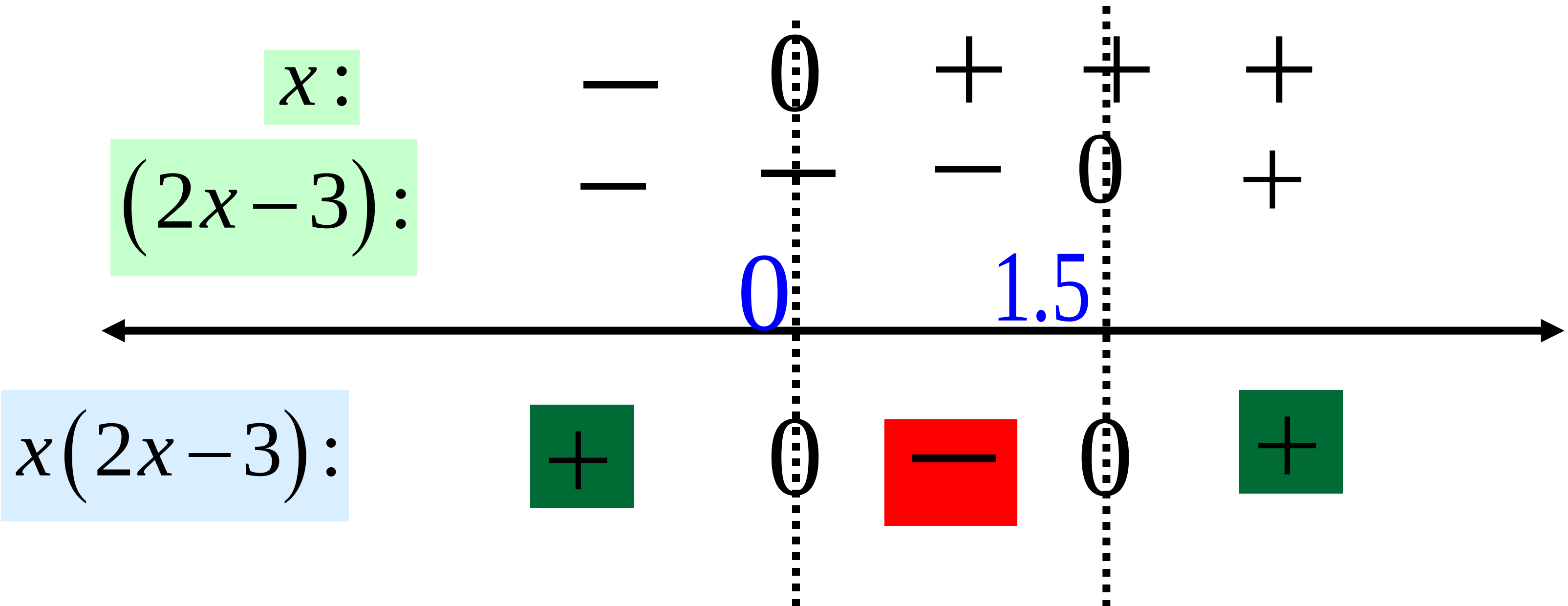
1. Determine the values of x such that

$$2x^2 - 3x \leq 0$$

Solution:

$$2x^2 - 3x < 0 \Rightarrow x(2x-3) < 0$$

$$\Longrightarrow \boldsymbol{x} \in \left(\mathbf{0}; \frac{\mathbf{3}}{\mathbf{2}}\right)$$

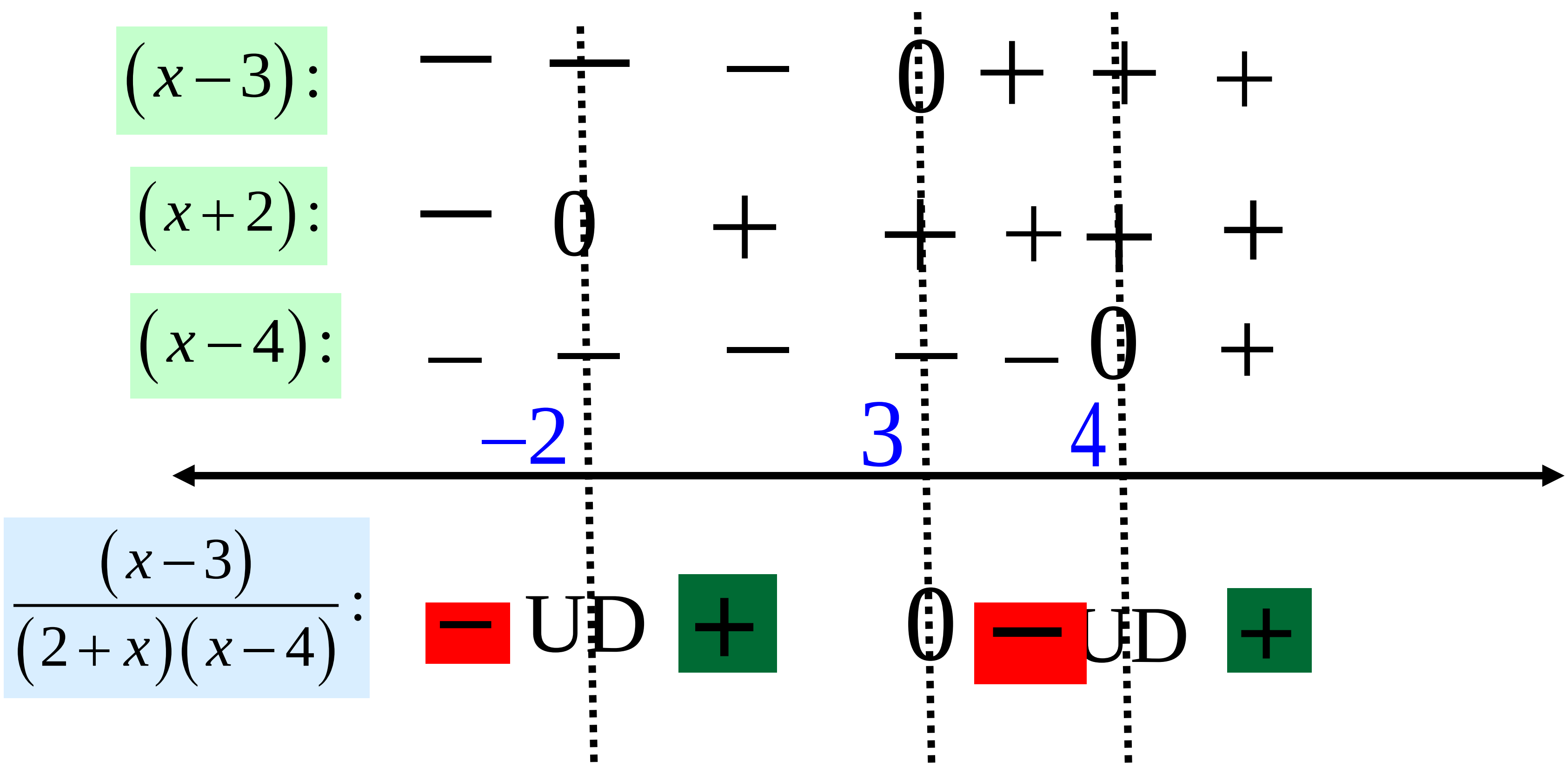


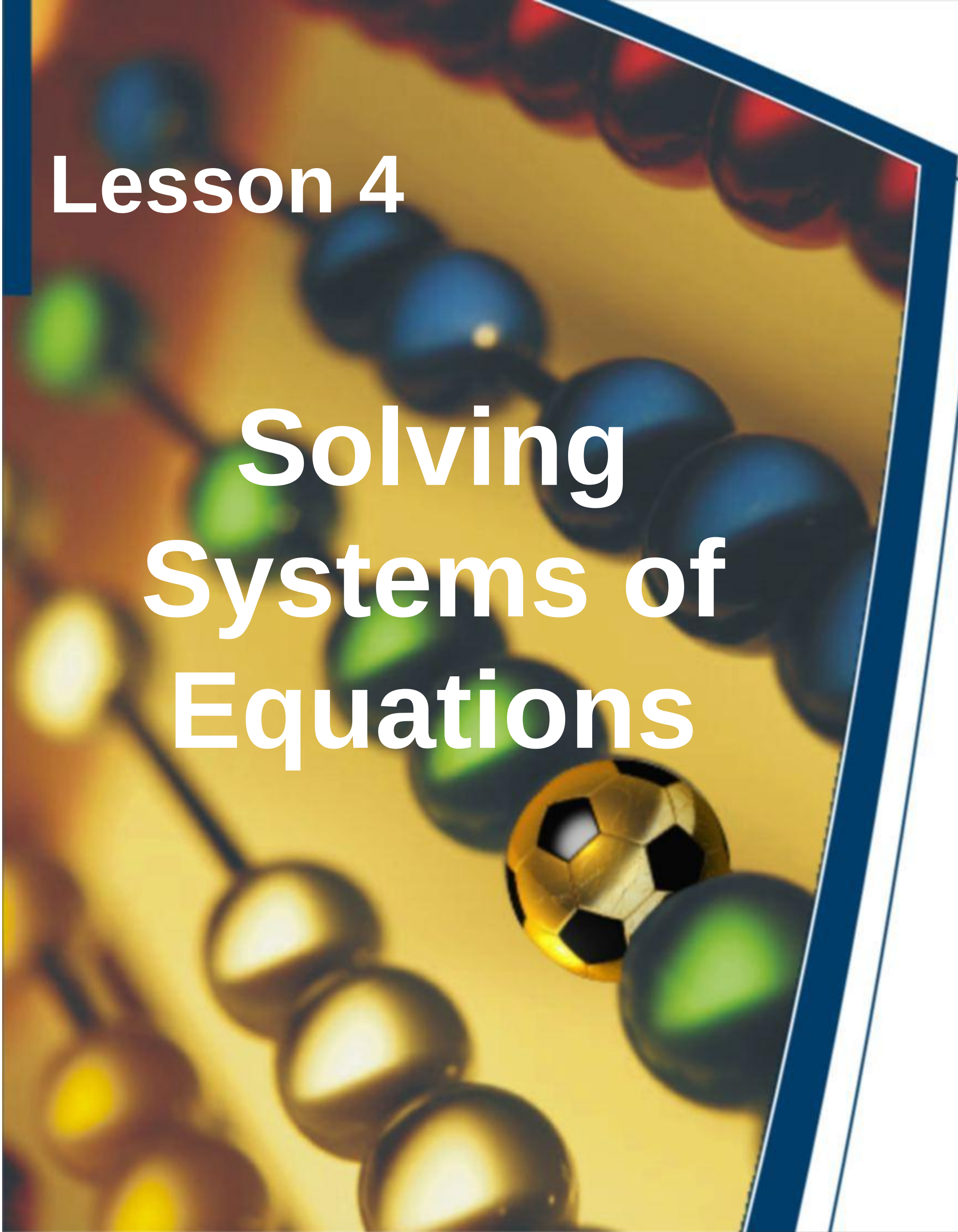
Tutorial 5 Problem 2: Suggested Solution

2. Determine the values of x such that

$$\frac{(x-3)}{(2+x)(x-4)} \leq 0$$

$$\Rightarrow x \in (-\infty; -2) \cup [3; 4)$$





Lesson 4

Solving Systems of Equations

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Simultaneous Algebraic Solution of two Linear Equations in Two Unknowns

Solve for x and y simultaneously:

$3x-2y=1 \cdots (1)$ and $x-y=2 \cdots (2)$

Make x the subject of the formula in (2):

So we have: $x=y+2$

Now **substitute** this value for x in (1):

So $3(y+2)-2y=1$

$$\Leftrightarrow 3y+6-2y=1 \Leftrightarrow y=-5$$

So $x=-5+2=-3$

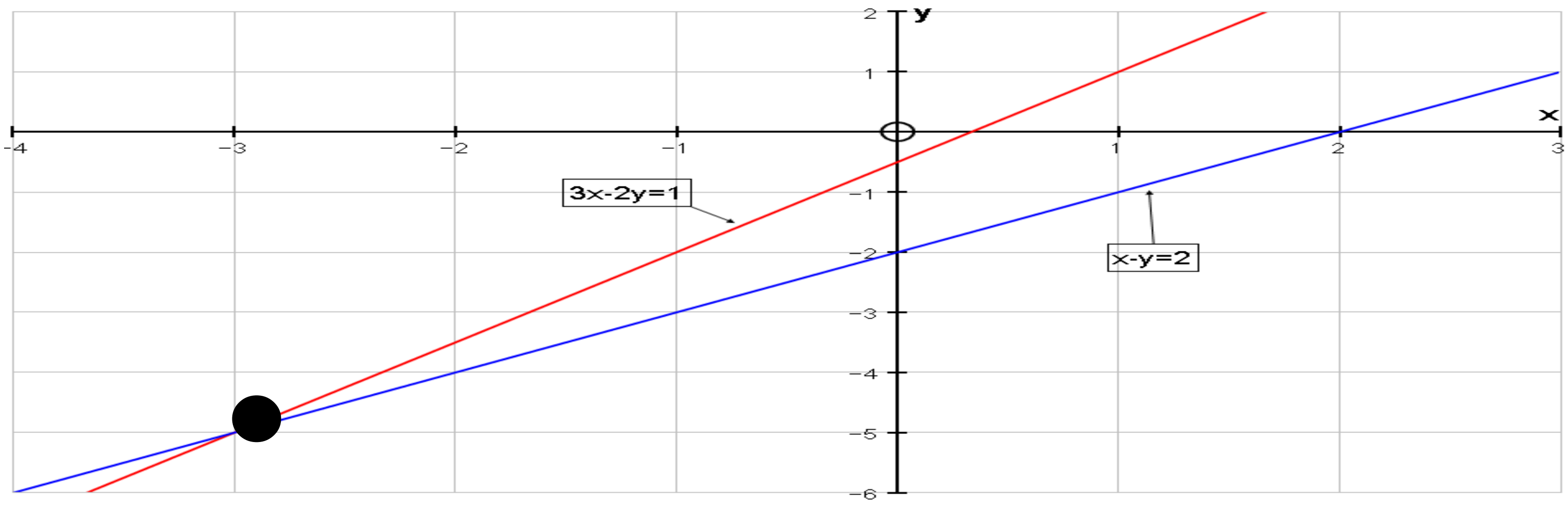
$$\therefore (x;y) \in \{(-3;-5)\}$$

Graphical Solution of two Simultaneous Linear Equations in Two Unknowns

From previous slide we know that:

$$\{(x;y): 3x-2y=1\} \cap \{(x;y): x-y=2\} = \{(-3;-5)\}$$

This implies that the two straight line intersect in the point $(-3;-5)$ or have the point $(-3;-5)$ in common.



Solving Simultaneously a Linear and a Quadratic equation in Two Unknowns

We can now use similar methods in solving 2 equations simultaneously, where one is linear and the other is quadratic

Solve for x and y if $x - y = 3 \cdots (1)$ and $x^2 + y = 27 \cdots (2)$

From (1): $y = x - 3 \cdots (3)$

Substitute (3) into (2):

$$\therefore x^2 + (x-3) = 27 \Rightarrow x^2 + x - 30 = 0$$

$$\therefore (x+6)(x-5) = 0 \Rightarrow x = -6 \text{ or } x = 5$$

Back substitute $x -$ values into (3):

$$\Rightarrow y = -6 - 3 = -9 \text{ or } y = 5 - 3 = 2$$

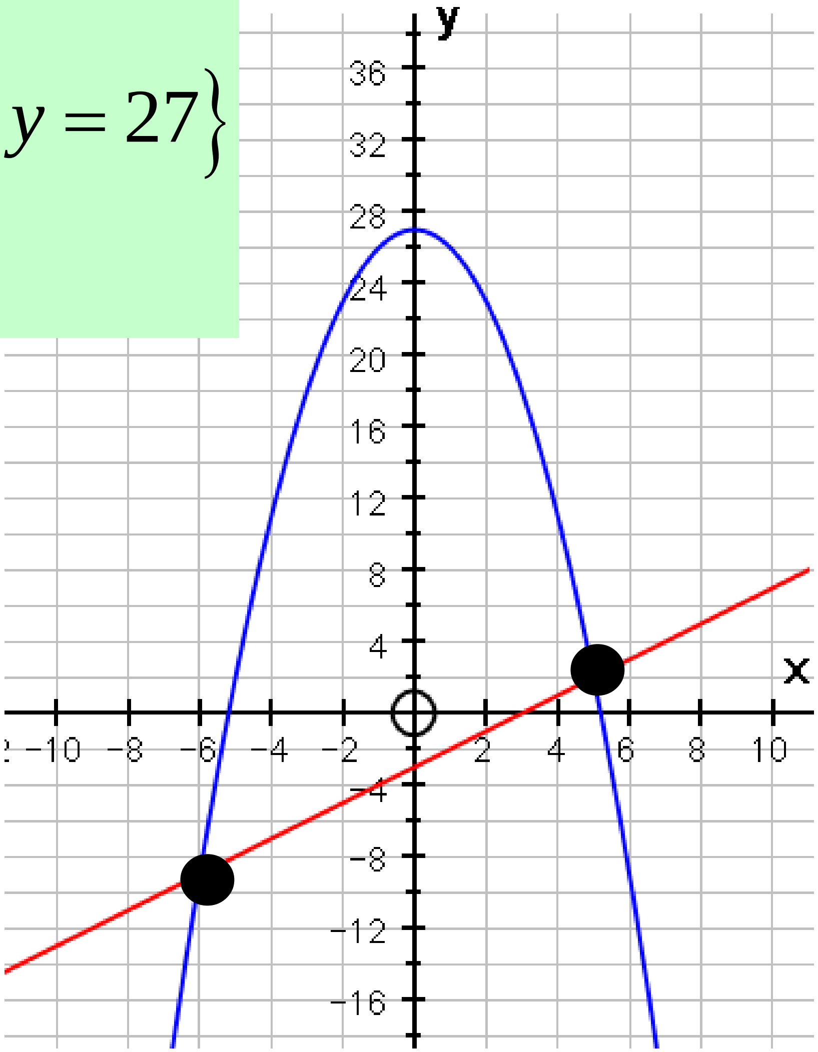
$$\therefore \text{ Solution Set } = \{(-6;-9);(5;2)\}$$

Solving Simultaneously a Linear and a Quadratic Equation Graphically

From previous slide:

$$\{(x;y): x-y=3\} \cap \{(x;y): x^2+y=27\}$$

$$=\{(-6;-9);(5;2)\}$$



Tutorial 6 : Solving Systems of Equations

Solve the following systems of simultaneous equations:

1. $y = x^2 - 3$

   $y = 2x - 4$ (algebraicly)

2. $y = -x^2 + 4$

   $y = x + 2$ (graphically)

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- Do Tutorial 6
- Then View Solutions

Tutorial 6 Problem 1: Suggested Solution

1. $y = x^2 - 3 \qquad \rightarrow (A)$

$y = 2x - 4 \qquad \rightarrow (B)$

Substitute for y in equation (A)

We have: $2x - 4 = x^2 - 3$

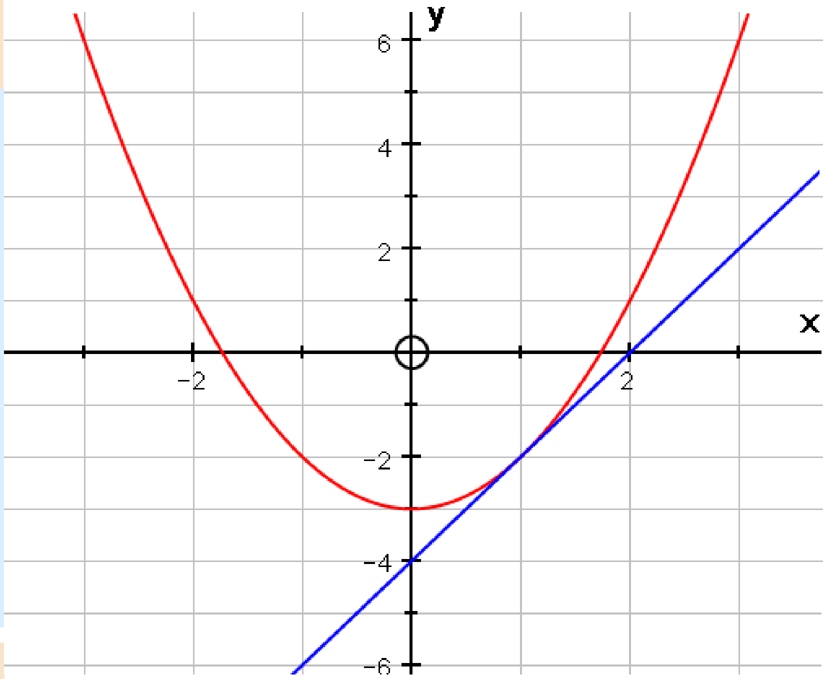
$\Leftrightarrow x^2 - 2x + 1 = 0$

$\Leftrightarrow (x-1)(x-1) = 0$

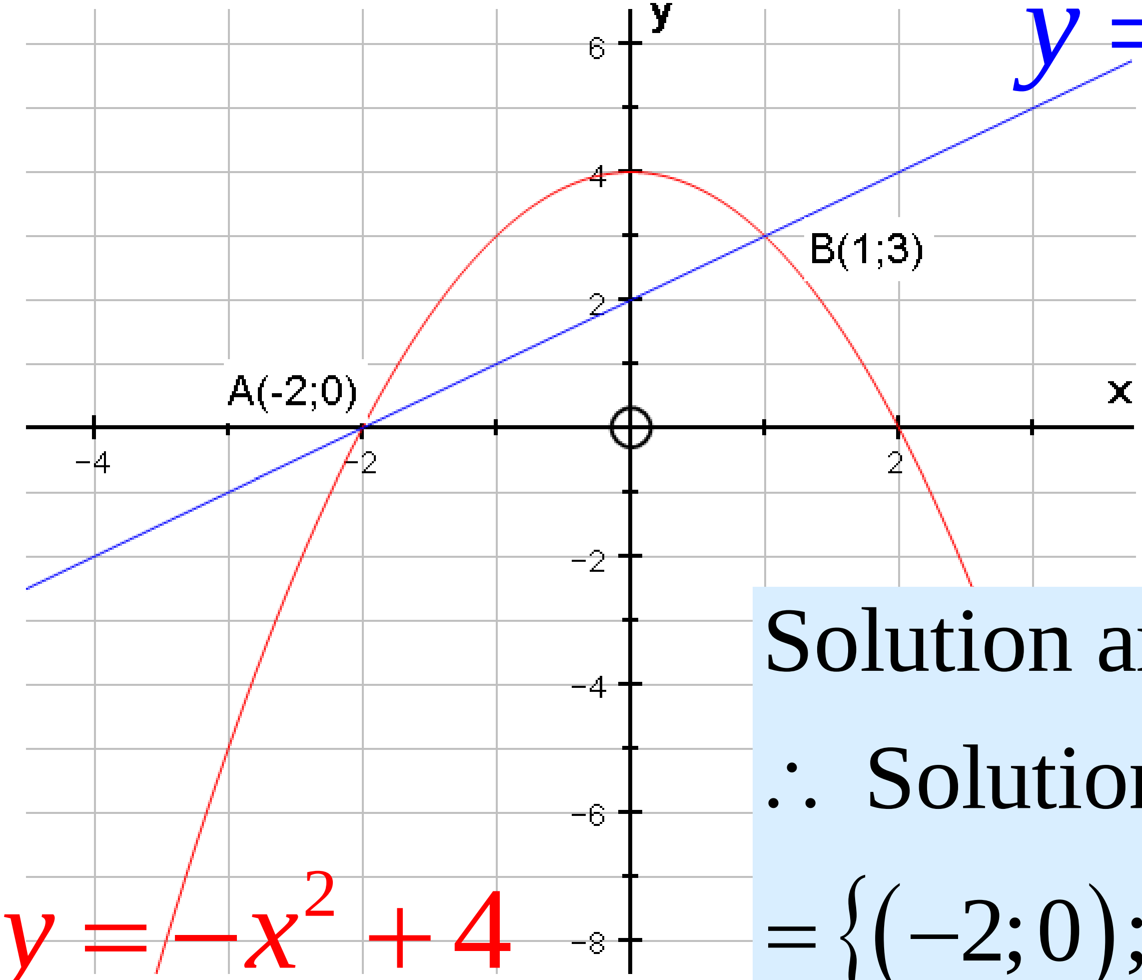
$\Leftrightarrow x = 1$ (only)

Hence, $y = 2(1) - 4 = -2$

Solution Set $\{(1;-2)\}$



Tutorial 6 Problem 2: Suggested Solution



$y = x + 2$

$y = -x^2 + 4$

Solution are points *A* and *B*.

$\therefore$ Solution Set

$= \{(-2;0);(1;3)\}$

End of the DVD on Solving Equations and Inequalities

REMEMBER!

- Consult text-books for additional examples.
- Attempt as many as possible other similar examples on your own.
- Compare your methods with those that were discussed in the DVD.
- Repeat this procedure until you are confident.
- Do not forget:

Practice makes perfect!