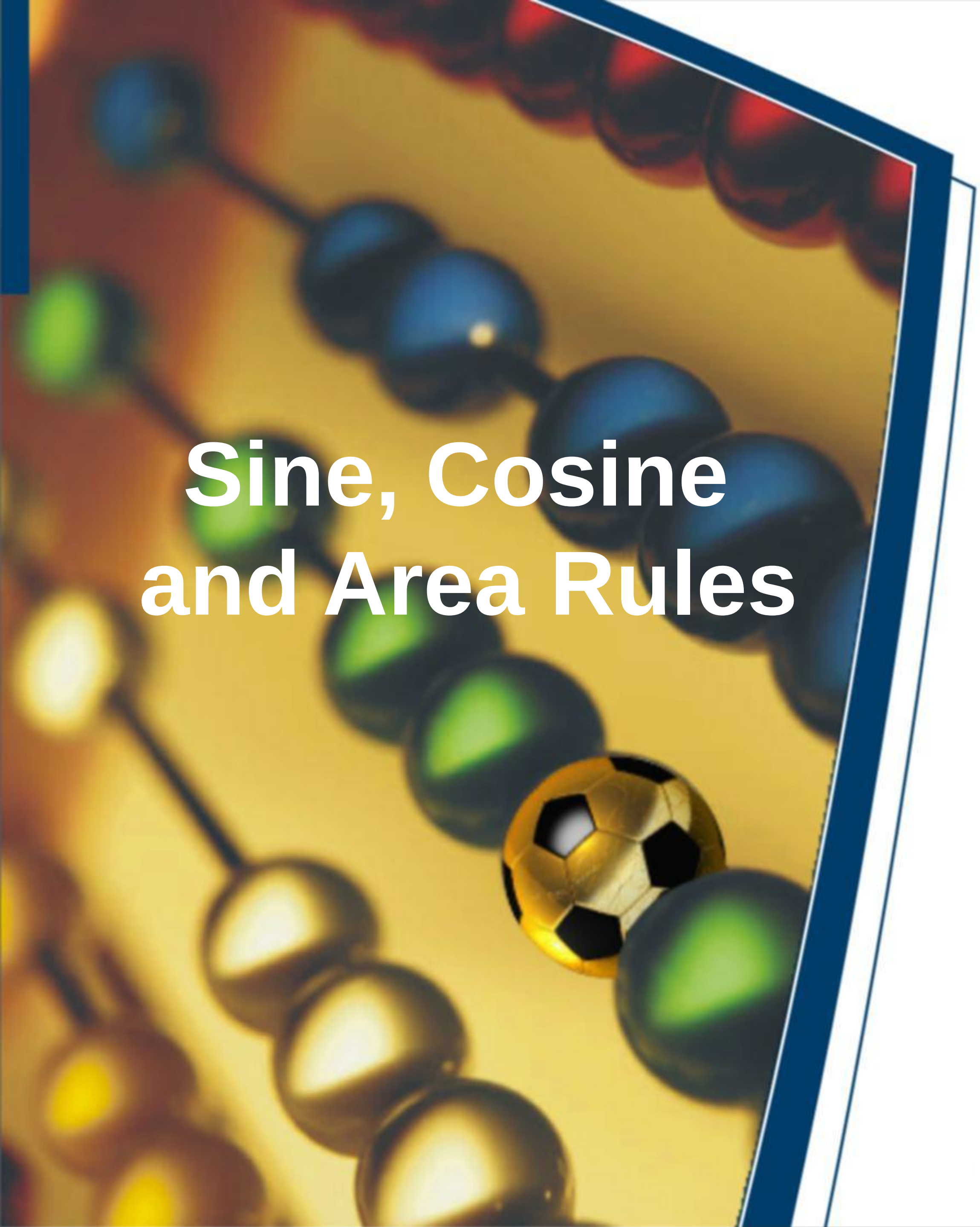

Sine, Cosine and Area Rules

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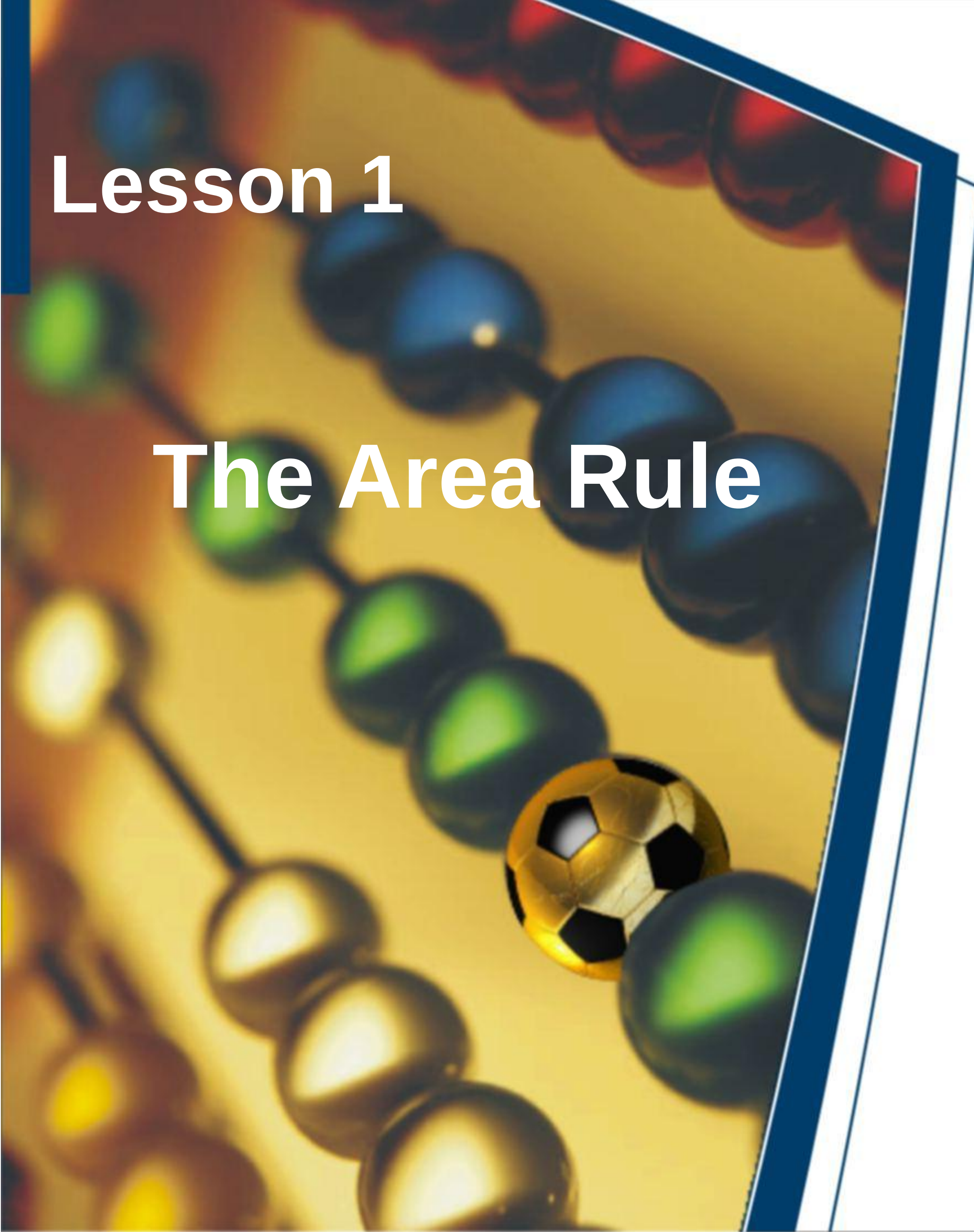
Outcomes for this DVD

In this DVD we will:

1. Calculate the area of a triangle given an angle and the two adjacent sides. Lesson 1
2. Apply the Sine Rule for triangles to calculate an unknown side or an unknown angle of a given triangle. Lesson 2
3. Apply the Cosine Rule for triangles to calculate an unknown side or an unknown angle of a given triangle. Lesson 3
4. Apply the Sine and the Cosine rules to solve problems in 2-dimensions. Lesson 4



Lesson 1

The Area Rule

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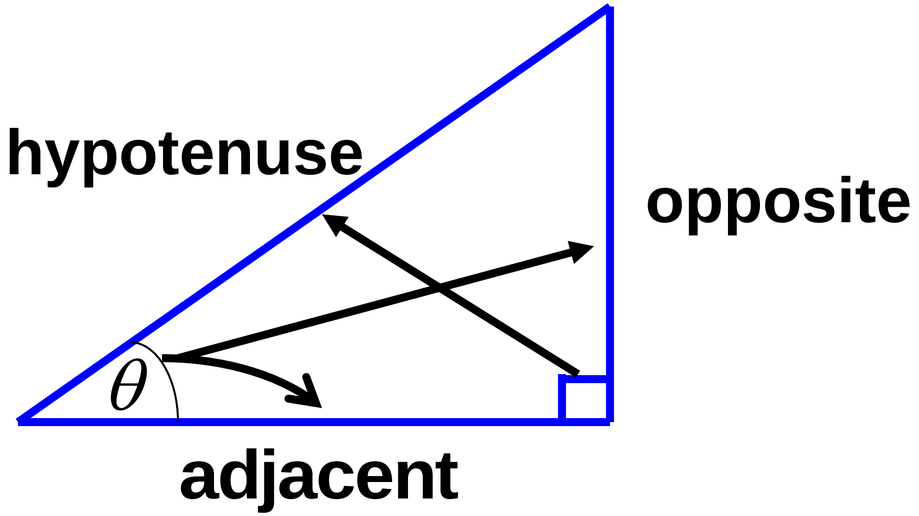
Some basic definitions – a reminder

Trigonometric Ratios

In a **right angled triangle**, the 3 trigonometric ratios for an angle θ are defined as follows:



$$\sin\theta = \frac{\textbf{opposite}}{\textbf{hypotenuse}}$$

$$\tan\theta = \frac{\textbf{opposite}}{\textbf{adjacent}}$$

$$\cos\theta = \frac{\textbf{adjacent}}{\textbf{hypotenuse}}$$

The area formula of a triangle

Consider a non-right angled triangle ABC.

a, b and c are the sides opposite angles A, B and C respectively.

(This is the conventional way of labelling a triangle).

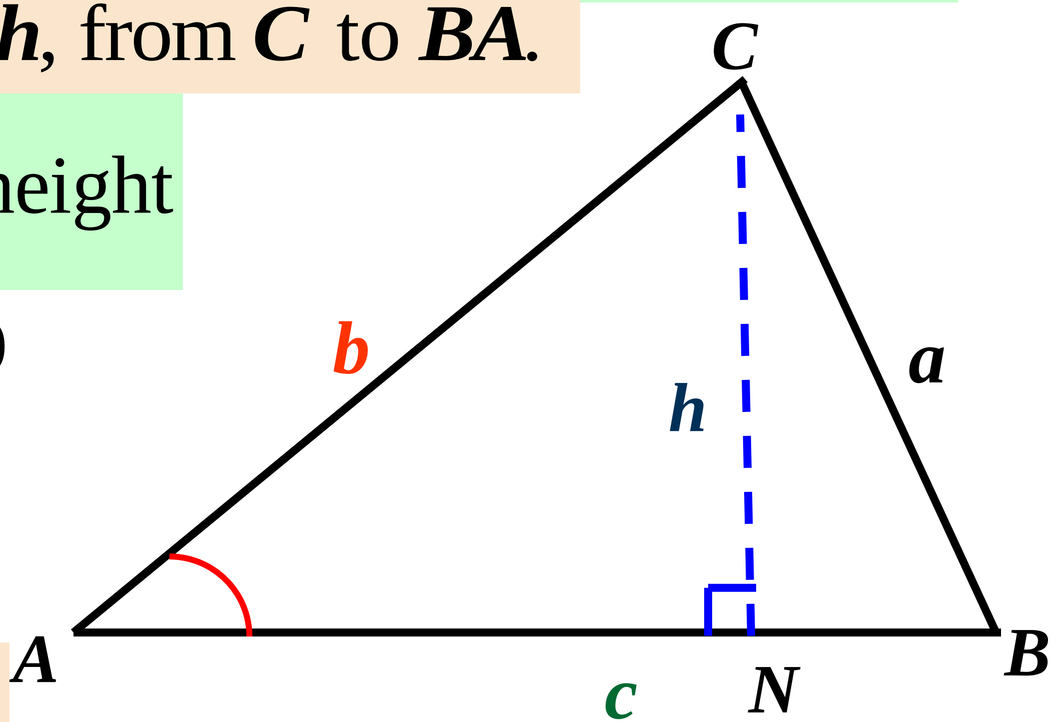
Draw the perpendicular, h, from C to BA.

$$\text{Area of } \Delta = \frac{1}{2}\text{base} \times \text{height}$$

$$\Rightarrow \quad \textbf{Area} = \tfrac{1}{2} c \times h \quad \text{--- (1)}$$

In ΔACN, $\sin A = \dfrac{h}{b}$

$$\Rightarrow \quad b \sin A = h$$



Substituting for h in (1)

$$\Rightarrow \textbf{Area} = \tfrac{1}{2} c \times b \sin A$$

$$\textbf{Area} = \tfrac{1}{2} bc \sin A$$

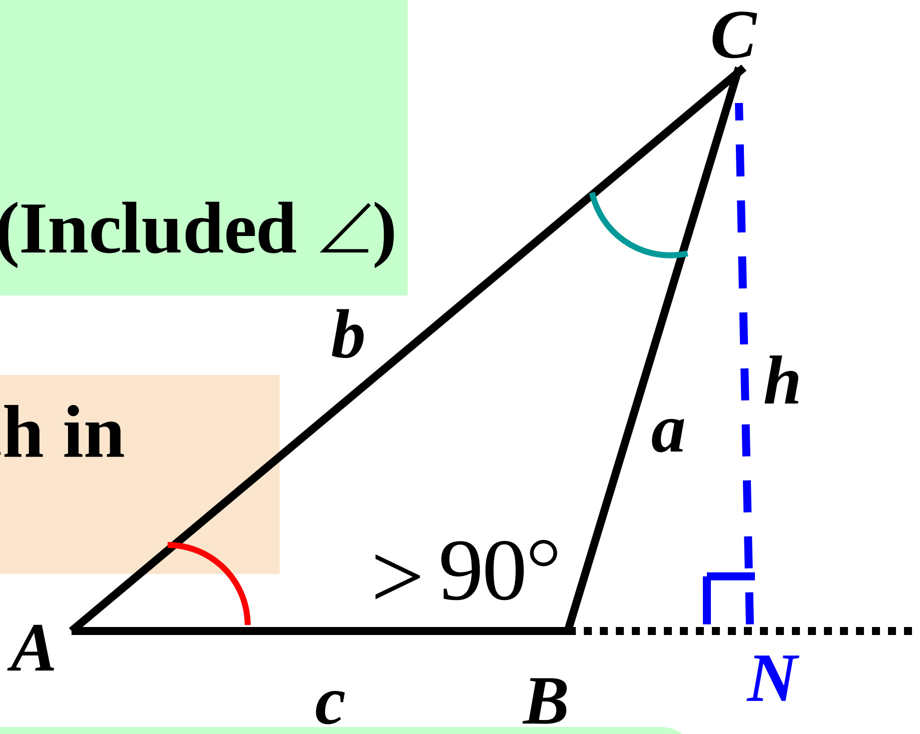
Different forms of the area formula

A similar argument gives the same formula for the area if $\angle B$ is obtuse i.e. $\angle B > 90°$

The formula always uses

- **<u>2 sides</u> and the**
- **<u>angle formed by those sides</u> (Included $\angle$)**

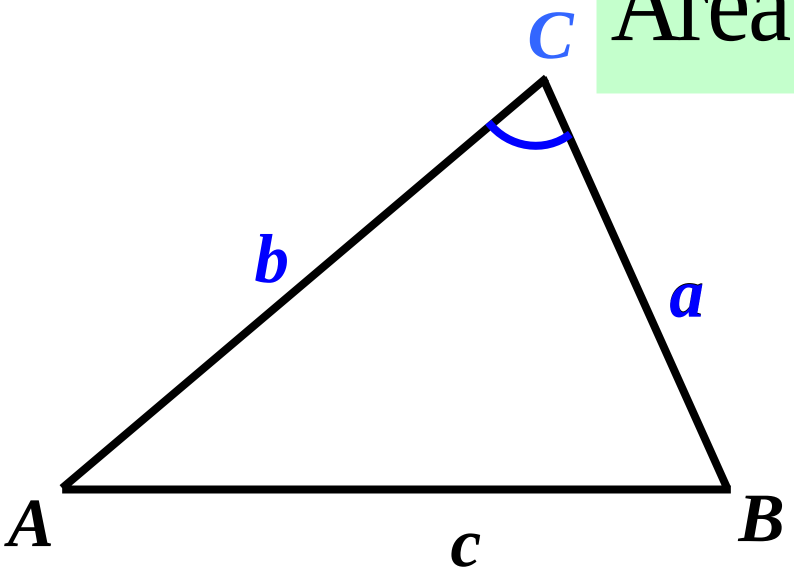
Any angle can be used as such in area formula, so



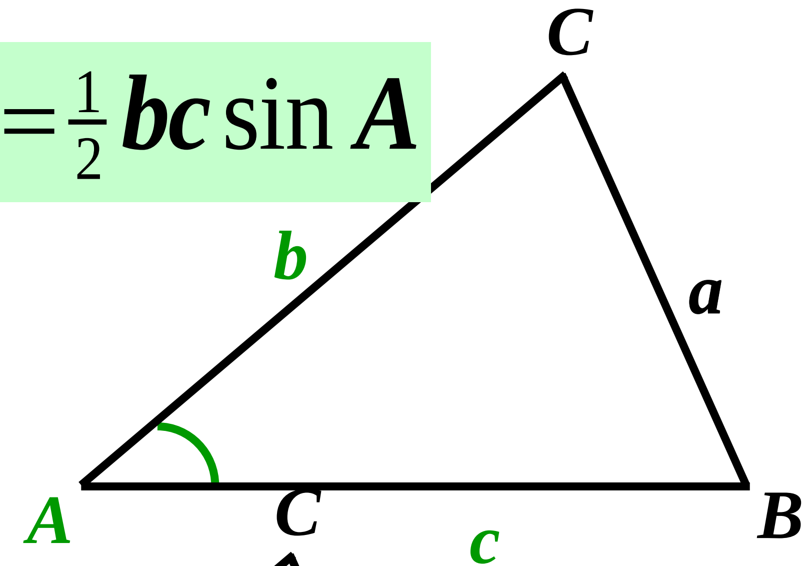
Area = $\frac{1}{2}ab\sin C$ **=** $\frac{1}{2}bc\sin A$ **=** $\frac{1}{2}ca\sin B$

Three possible approaches to find the area of a triangle

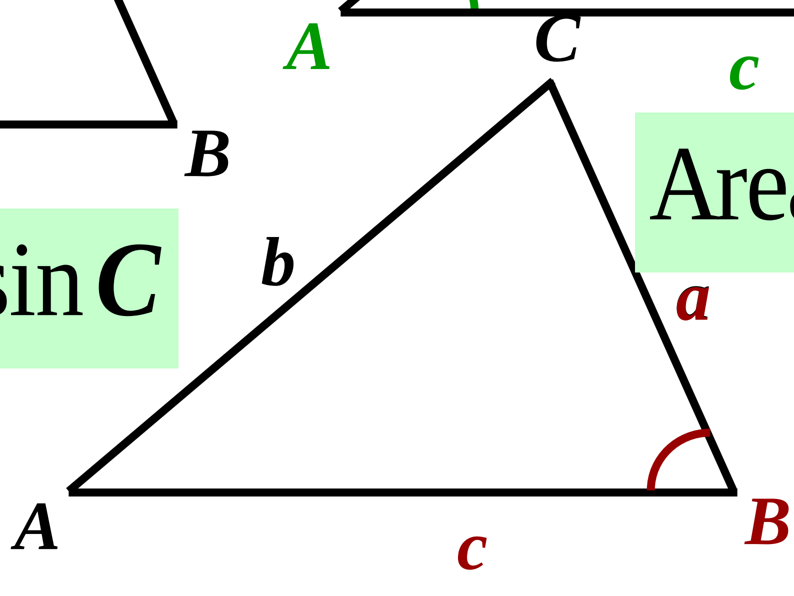
Any angle can be used in the formula, so



$$\text{Area} = \tfrac{1}{2}\boldsymbol{ab}\sin \boldsymbol{C}$$



$$\text{Area} = \tfrac{1}{2}\boldsymbol{bc}\sin \boldsymbol{A}$$



$$\text{Area} = \tfrac{1}{2}\boldsymbol{ca}\sin \mathbf{B}$$

The area of a triangle – Example 1

Find the area of ΔPQR.

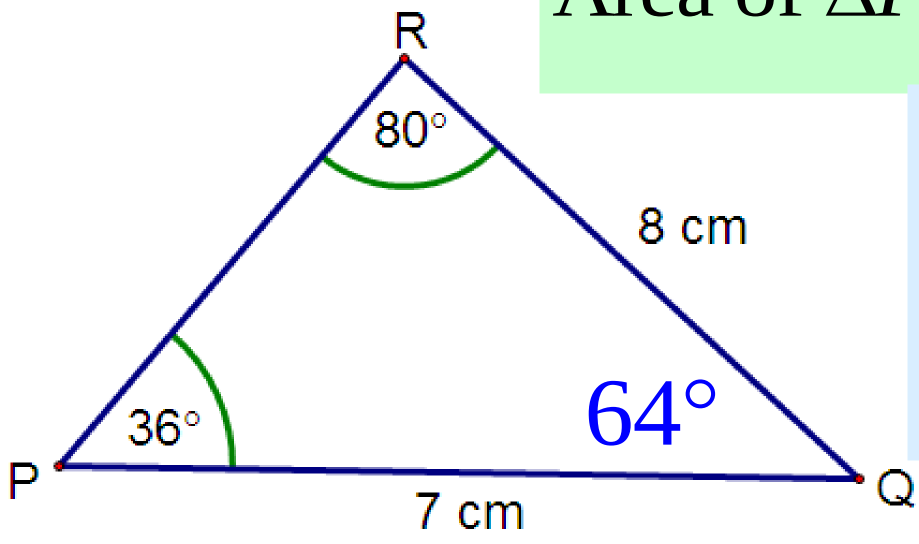
Solution: We must use the angle formed by the 2 sides with the given lengths.

We know PQ and RQ so use the included angle Q

$$\text{Area of } \Delta PQR = \frac{1}{2} \times QP \times QR \times \sin Q$$

$$= \frac{1}{2} \times 8 \times 7 \times \sin 64^\circ \text{ cm}^2$$

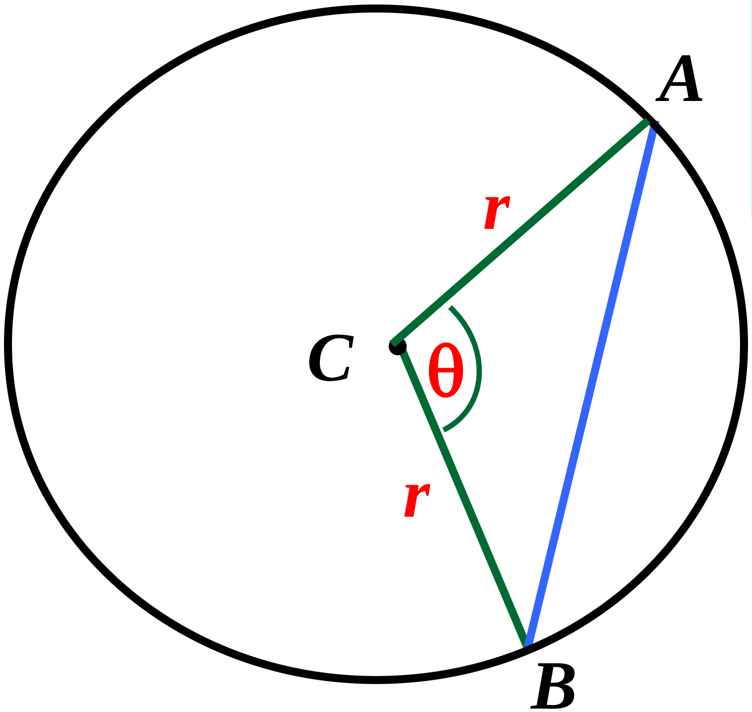
$$= 25,2 \text{ cm}^2$$



The area of a triangle – Example 2

Find the area of $\triangle ABC$.

A useful application of the area formula occurs when we have a triangle formed by 2 radii and a chord of a circle.



$$\text{Area} = \frac{1}{2}(CA)(CB)\sin C$$

$$\text{But } CA = CB = r$$

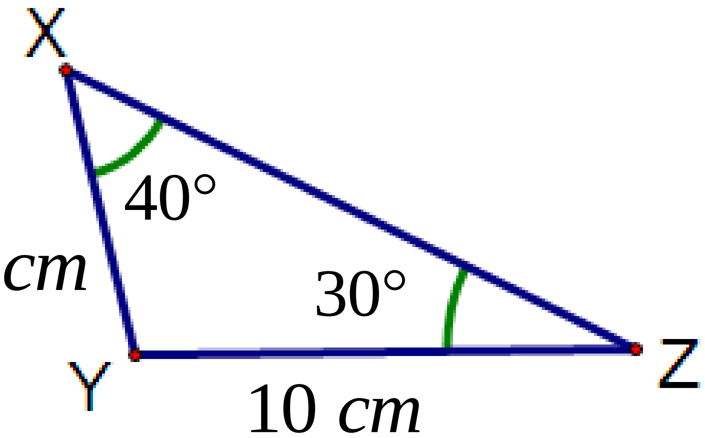
$$\therefore \textbf{Area} = \frac{1}{2}\boldsymbol{r}^2 \sin\theta$$

Tutorial 1: Area of a Triangle

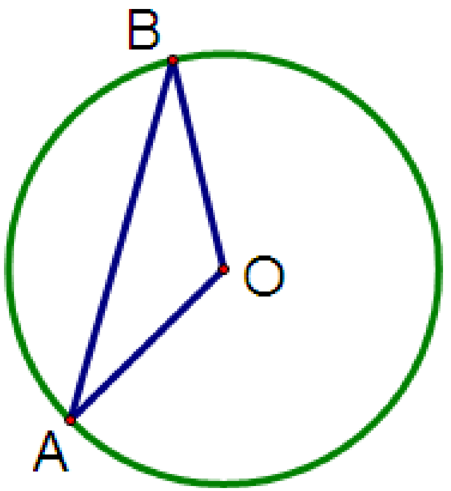
Find the areas of the triangles shown in the diagrams.

Give your answers accurate to 2 decimal digits

1)



2)



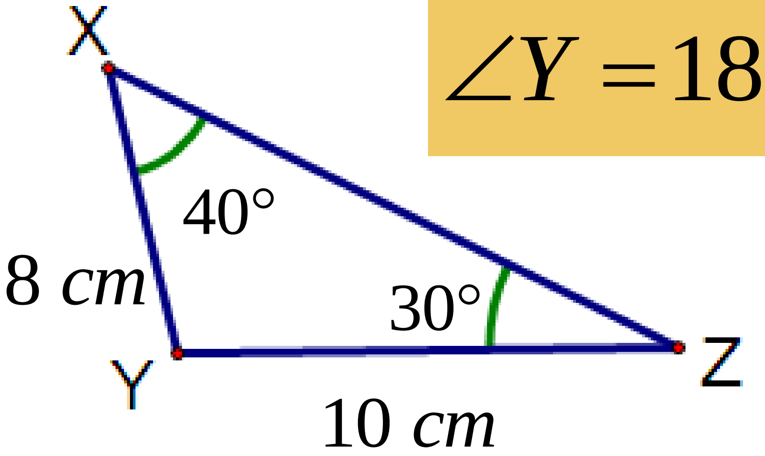
$\text{radius} = 6\ cm$

$\angle AOB = 120°$

PAUSE

- Do Tutorial 1
- Then View Solutions

Tutorial 1: Problem 1: Area of a Triangle: Solution



$$\angle Y = 180° - (40° + 30°) = 110°$$

$$1)\quad \text{Area } \Delta XYZ = \frac{1}{2} \times XY \times YZ \times \sin Y$$

$$= \frac{1}{2} \times z \times x \times \sin Y$$

$$= \frac{1}{2} \times 8 \times 10 \times \sin(110°)\ \text{cm}^2$$

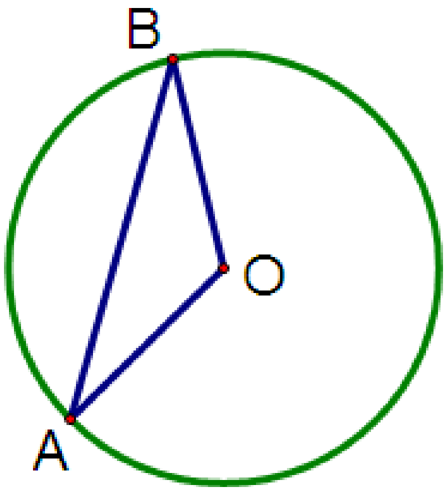
$$= 37{,}59\ \text{cm}^2$$

Tutorial 1: Problem 2: Area of a Triangle: Solution

Given:

$$\text{radius} = 6\ cm$$

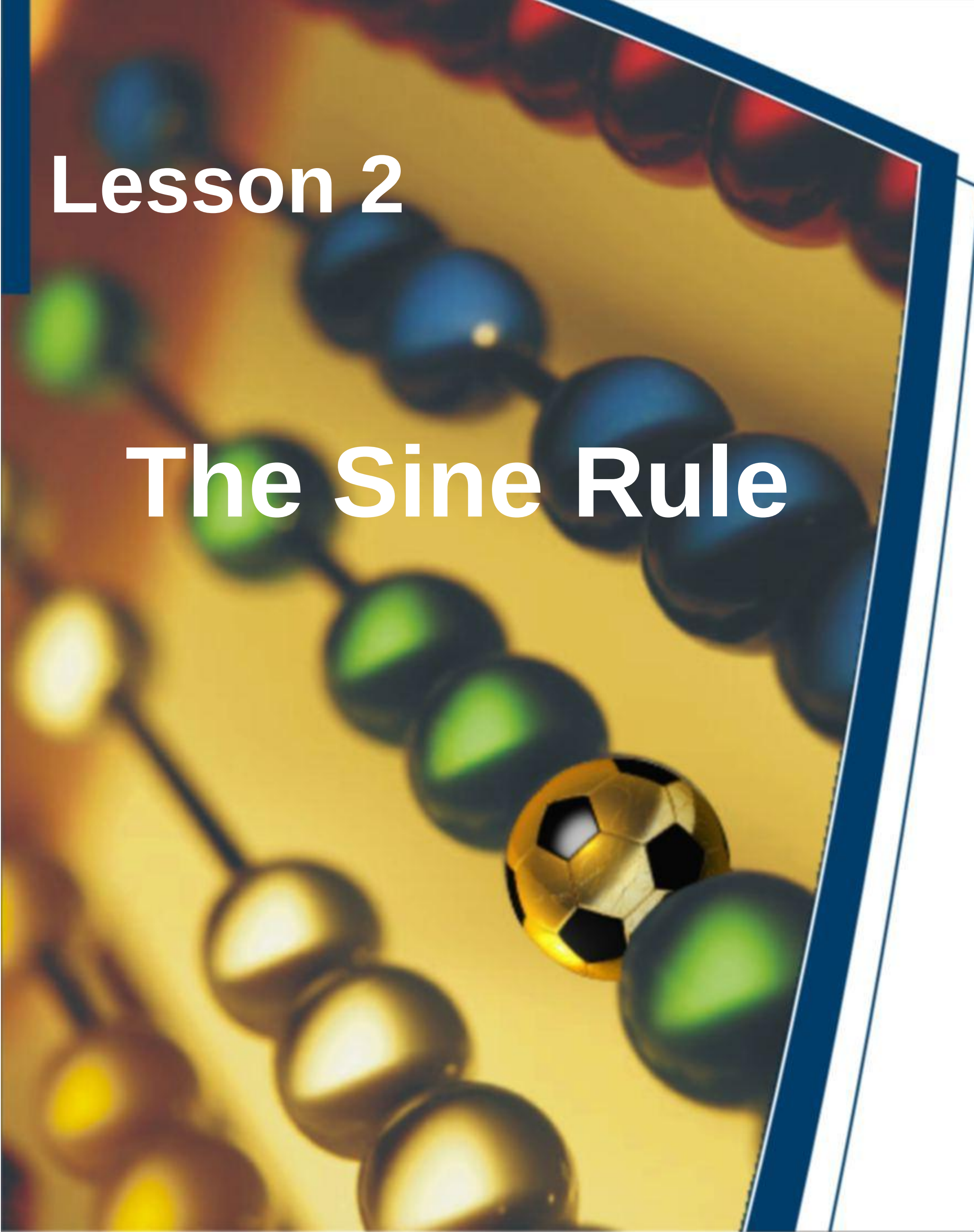
$$\angle AOB = 120^\circ$$



$$2)\quad \text{Area } \Delta AOB = \frac{1}{2} \times r^2 \times \sin O$$

$$= \frac{1}{2} \times (6\text{ cm})^2 \times \sin 120^\circ$$

$$= 15,59\text{ cm}^2$$



Lesson 2

The Sine Rule

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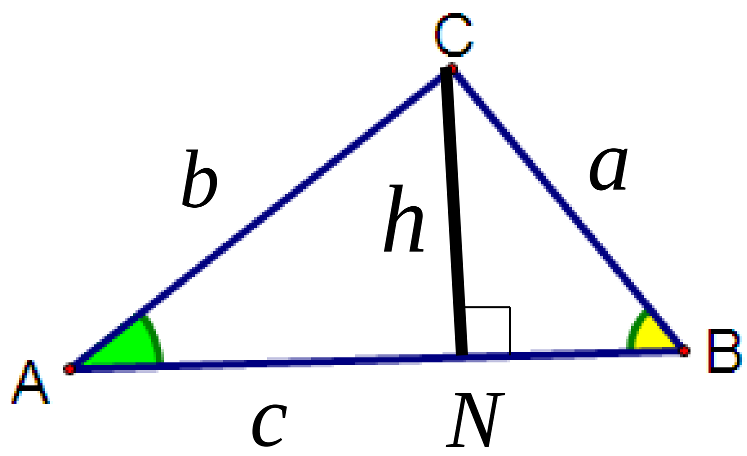
The Sine Rule for Triangles

One way to find unknown sides and angles in **non - right angled triangles** is by using the **Sine Rule**:

Suppose ΔABC is a scalene triangle

Drop $CN \perp AB$



In $\Delta ACN, \ \sin A = \dfrac{h}{b}$

$$\Rightarrow h = b \sin A$$

In $\Delta BCN, \quad \sin B = \dfrac{h}{a}$

$$\Rightarrow h = a \sin B$$

$$\therefore b \sin A = a \sin B$$

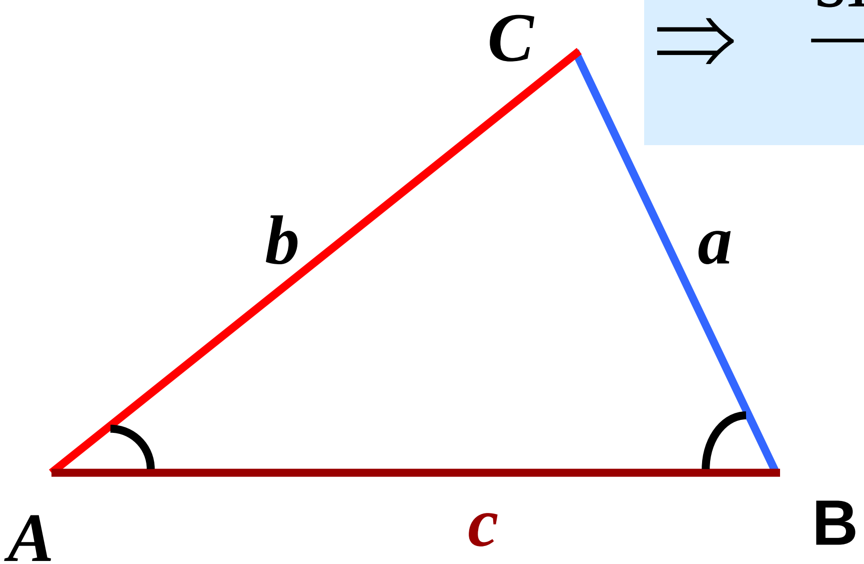
or $$\frac{\sin A}{a} = \frac{\sin B}{b}$$

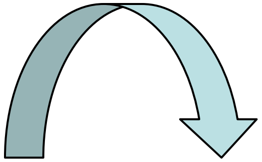
The Complete Sine Rule for Triangles

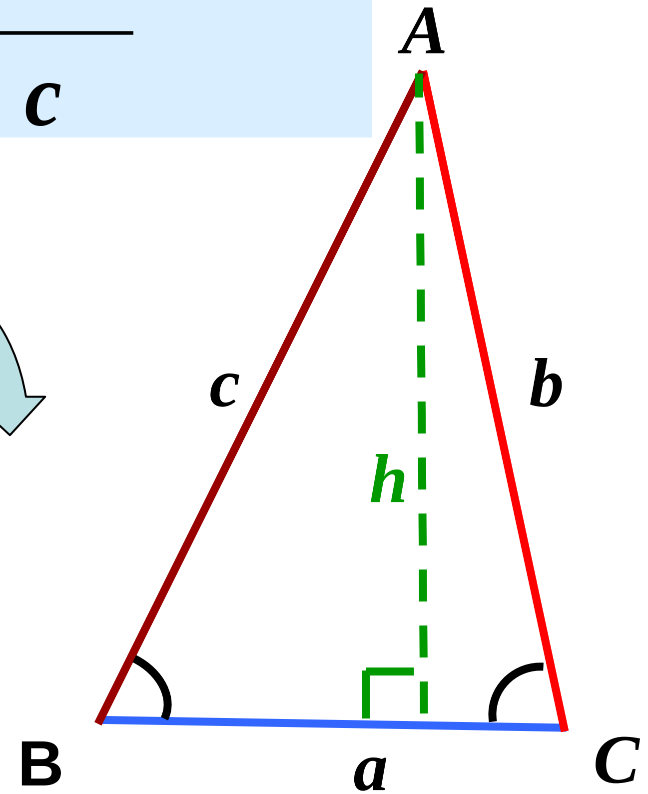
ΔABC can be turned so that BC is the base.
We then get

Now $h = c \sin B = b \sin C$

$$\Rightarrow \quad \frac{\sin B}{b} = \frac{\sin C}{c}$$





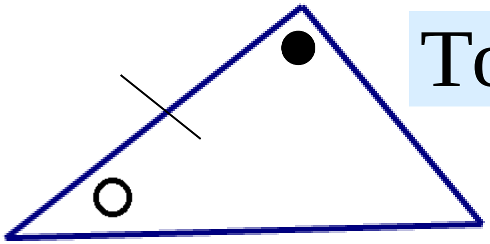


So $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

When do we use the Sine Rule?

The sine rule can be used in a triangle when:

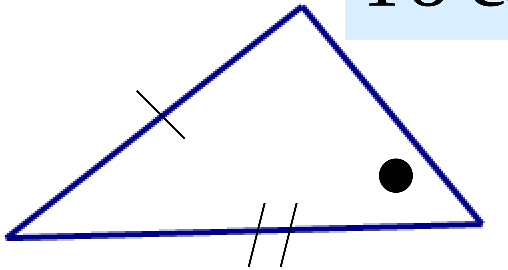
- Two angles and a side are given



To calculate second side

- Two sides and the non-included angle are given

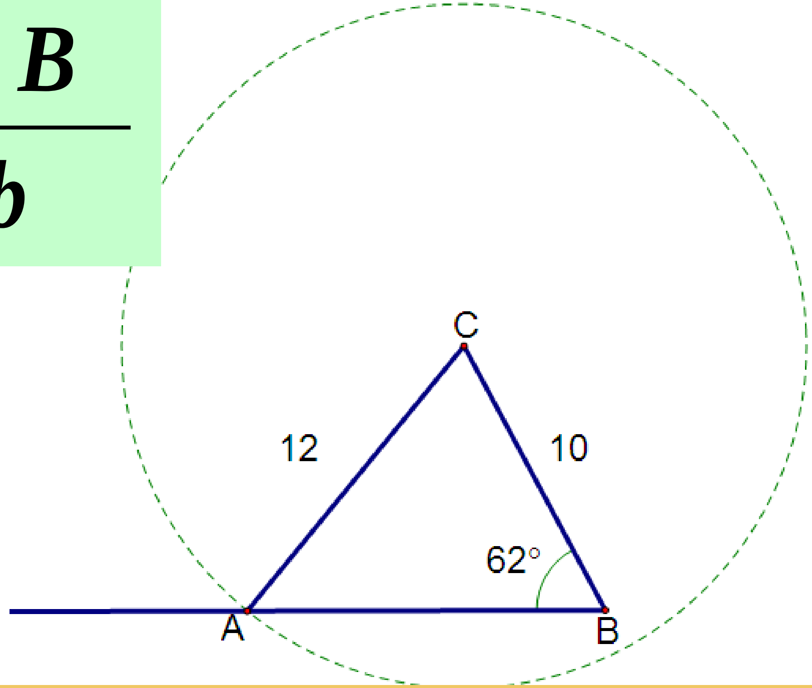
To calculate second angle



Application of the Sine Rule - Example 1

In ΔABC, find the size of angles A and C.

Solution: Use $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$



$$\Rightarrow \quad \sin A = \frac{a \sin B}{b}$$

$$\Rightarrow \quad \sin A = \frac{10 \sin 62^\circ}{12}$$

$$\Rightarrow \angle A = 47{,}4^\circ$$

$\angle A$ is opposite the shorter of the 2 given sides.

$\therefore \angle A < 62^\circ \Rightarrow \angle A$ must be an acute angle.

(Only one possibility as can be seen from sketch)

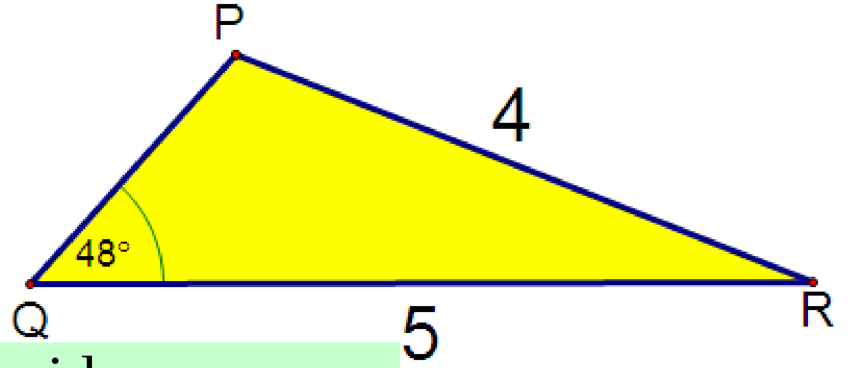
Thus $\angle C = 180^\circ - 62^\circ - 47{,}4^\circ = 70{,}6^\circ$

Application of the Sine Rule - Example 2

In ΔPQR it is given that:

$QR = 5,\ PR = 4$ and $\angle Q = 48°$.

Determine $\angle P$.



$\angle P$ is opposite the longer of the 2 given sides.

$\therefore \angle P > 48° \Rightarrow \angle P$ can be an acute or obtuse angle.

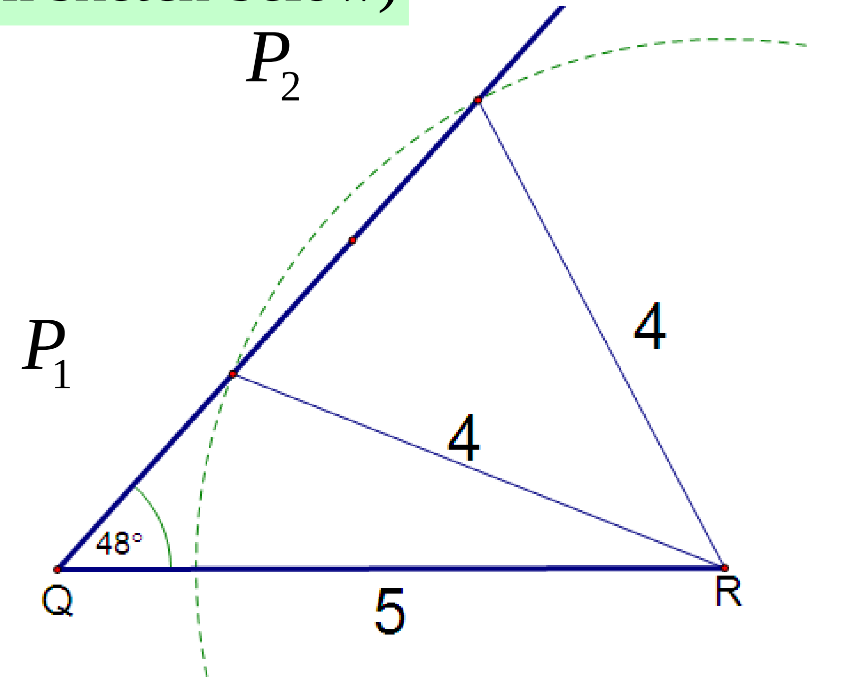
($\therefore$ Two possibilities as can be seen from sketch below)

Solution : Use $\dfrac{\sin Q}{q} = \dfrac{\sin P}{p}$

$$\Rightarrow \quad \sin P = \frac{p \sin Q}{q} = \frac{5 \times \sin 48°}{4}$$

$$\Rightarrow \quad \angle P_2 = 68,3° \ \textbf{or}$$

$$\angle P_1 = 180° - 68,3° = 111,7°$$



Application of the Sine Rule - Example 3

In $\triangle XYZ$, find the length XY.

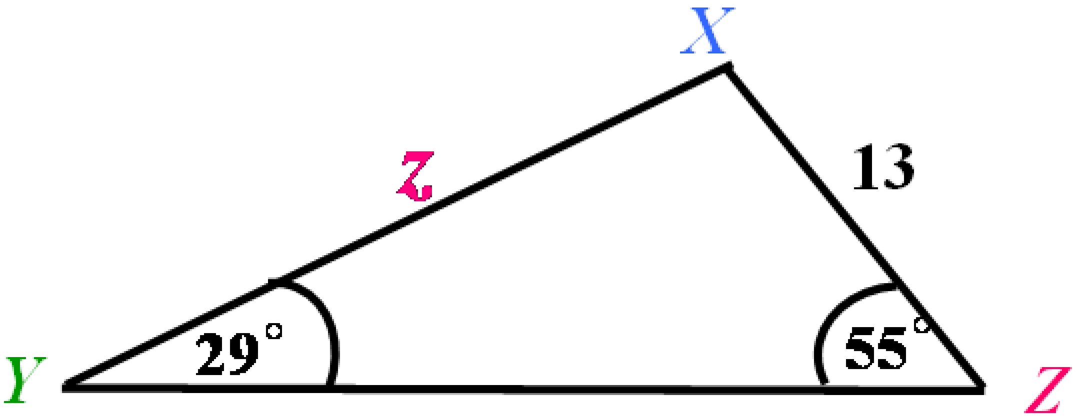
Solution : As the unknown is a side, we use the sine rule in its reciprocal form. The unknown side is then at the top.

$$\frac{z}{\sin Z} = \frac{y}{\sin Y}$$

$$\Rightarrow z = \frac{y \sin Z}{\sin Y}$$

$$\Rightarrow \quad z = \frac{13 \sin 55°}{\sin 29°}$$

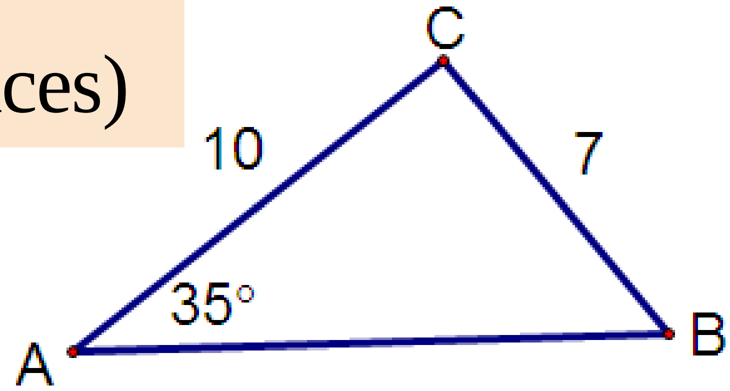
$$\Rightarrow \quad z = 22{,}0$$



Tutorial 2: Sine Rule

1. In ΔABC, find $\angle B$.

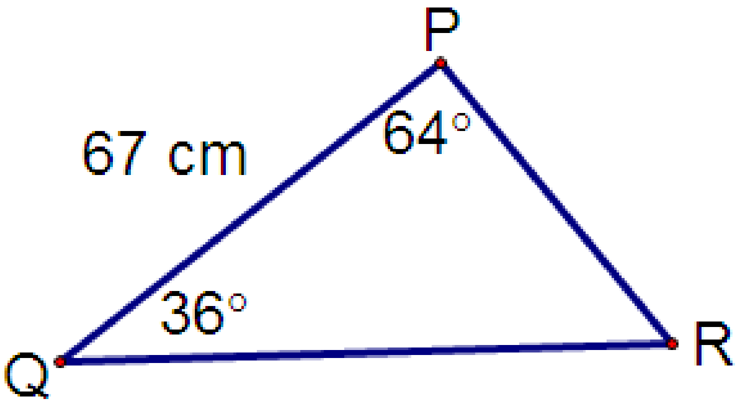
(Correct to two decimal places)



2. In ΔPQR, find QR and the area of ΔPQR

PAUSE DVD

- Do Tutorial 2
- Then View Solutions

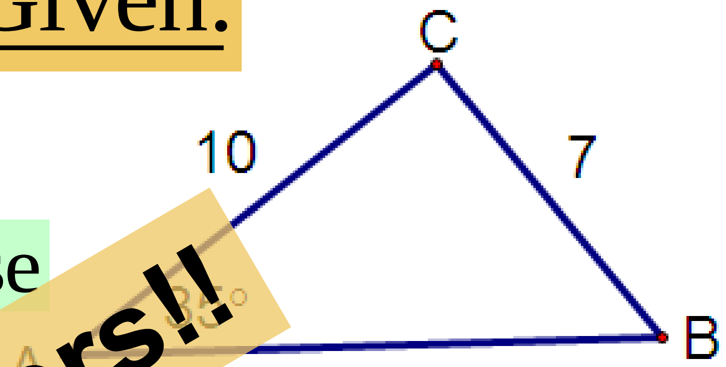


Tutorial 2: Problem 1: Sine rule: Solution

Find $\angle B$.

(2 decimal places)

Given:



$\angle B > 35° \Rightarrow \angle B$ acute or obtuse

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 35°}{7} = \frac{\sin B}{10}$$

$$\therefore \sin B = \frac{10 \sin 35°}{7}$$

$$\Rightarrow \angle B_1 = 55,02°$$

$$\text{or } \angle B_2 = 180° - 55,02° = 124,98°$$

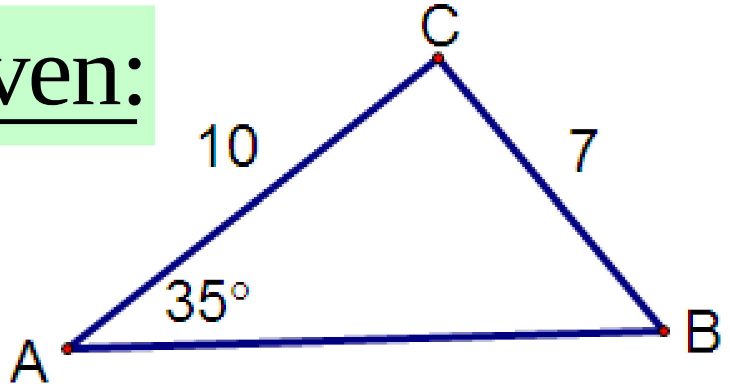
Two Possible Answers!!

Tutorial 2: Problem 1: Why two solutions?

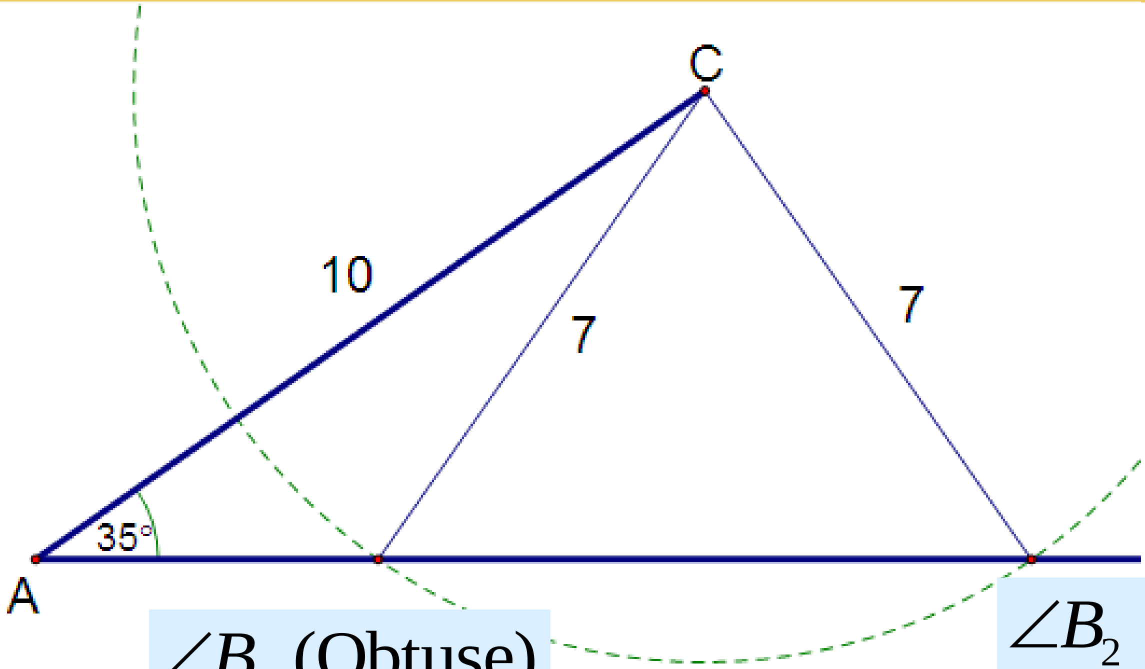
Obtained: $\angle B_1 = 55,02°$

or $\angle B_2 = 124,98°$

Given:



$$\angle B > 35° \Rightarrow \angle B \text{ acute or obtuse}$$



$\angle B_1$ (Obtuse)

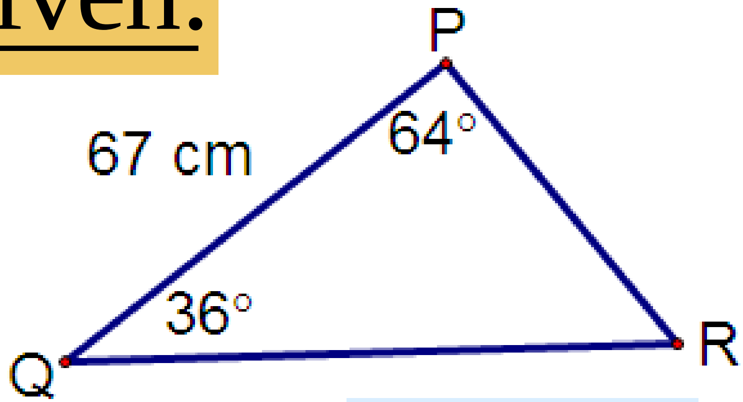
$\angle B_2$ (Acute)

Tutorial 2: Problem 2: Sine rule: Solution

2. Find QR and the area of ΔPQR.

$$\frac{QR}{\sin 64^\circ} = \frac{67}{\sin 80^\circ}$$

Given:



$$\therefore QR = \frac{67 \sin 64^\circ}{\sin 80^\circ} = 61{,}15 \text{ cm}$$

$$\angle R = 80^\circ$$

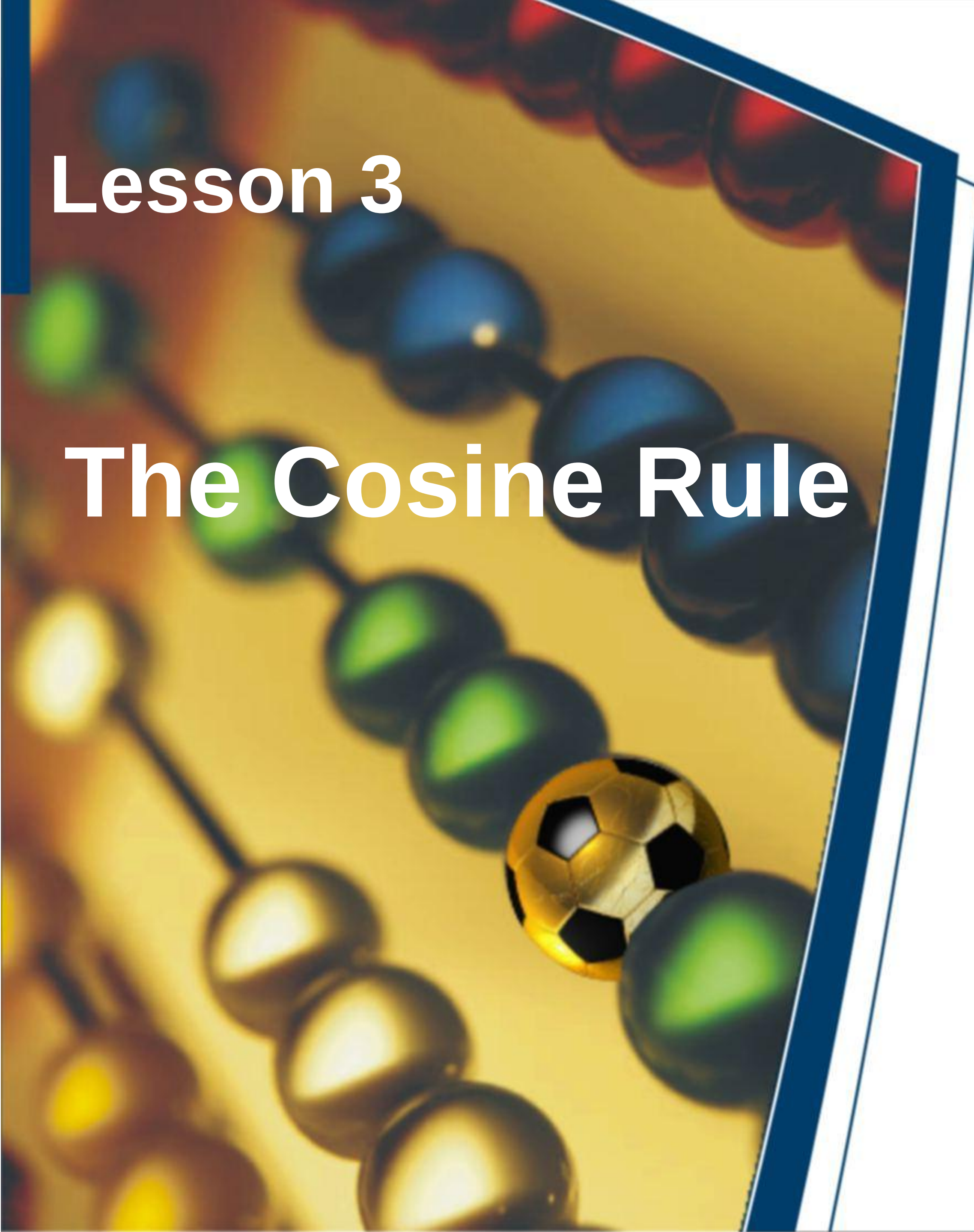
$$\text{Area of } \Delta PQR = \frac{1}{2} \times QP \times QR \times \sin 36^\circ$$

$$= \frac{1}{2} \times 67 \times 61{,}15 \times \sin 36^\circ$$

$$= 1\ 204{,}09 \text{ cm}^2$$

Lesson 3

The Cosine Rule



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The Cosine Rule for Triangles

The Cosine Rule for $\Delta \boldsymbol{ABC}$ is given by:

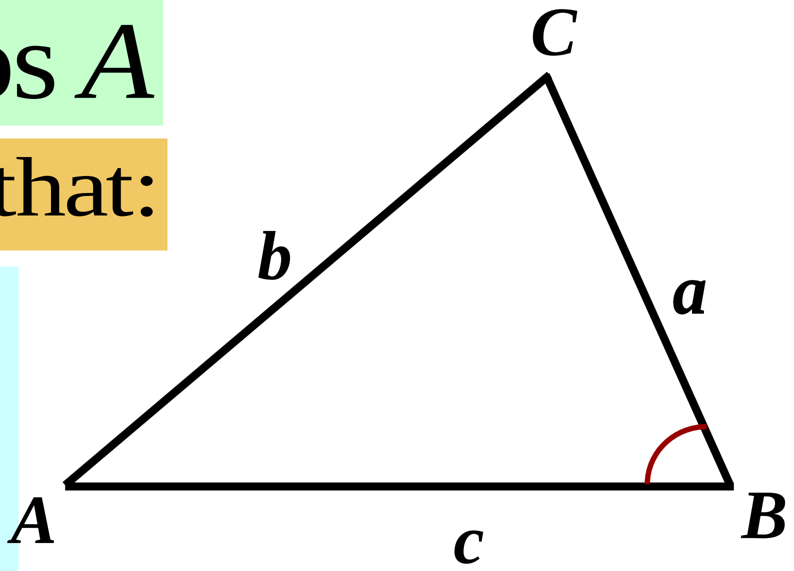
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Symmetry also implies that:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

or

$$c^2 = a^2 + b^2 - 2ab\sin C$$

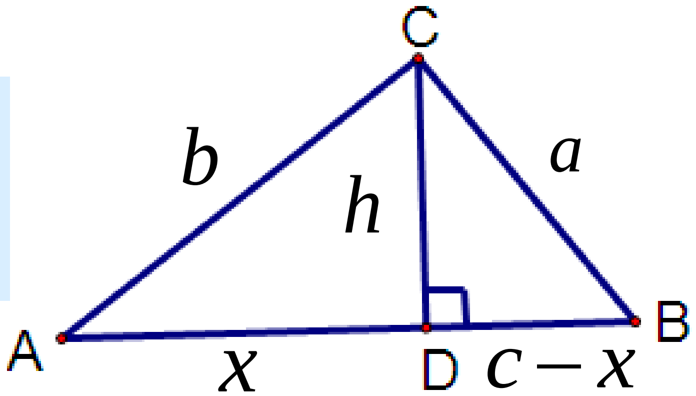


We use this form to find the third side when two sides and included angle are given.

Proof of the Cosine Rule

In ΔCAD:

$$\cos A = \frac{x}{b} \text{ and } b^2 = x^2 + h^2$$



In ΔBCD:

$$a^2 = h^2 + (c-x)^2 = h^2 + c^2 - 2cx + x^2$$

$$\therefore a^2 = \left(b^2 - x^2\right) + c^2 - 2c\left(b\cos A\right) + x^2$$

$$= b^2 + c^2 - 2bc\cos A$$

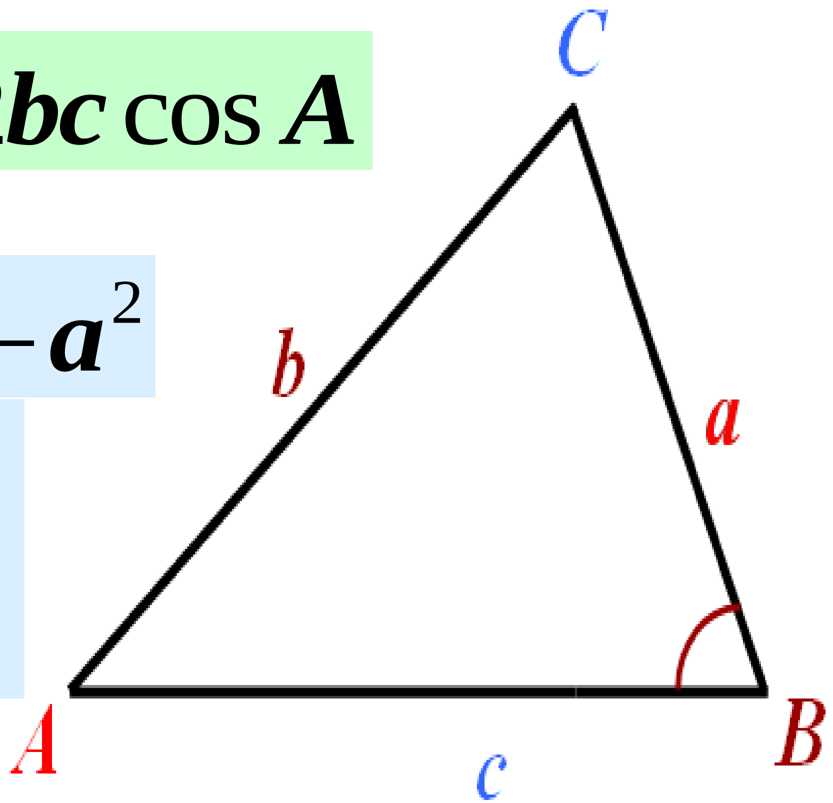
Proofs for symmetrical results are similar.

A second form of the Cosine Rule

Know: $a^2 = b^2 + c^2 - 2bc\cos A$

$\therefore \quad 2bc\cos A = b^2 + c^2 - a^2$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

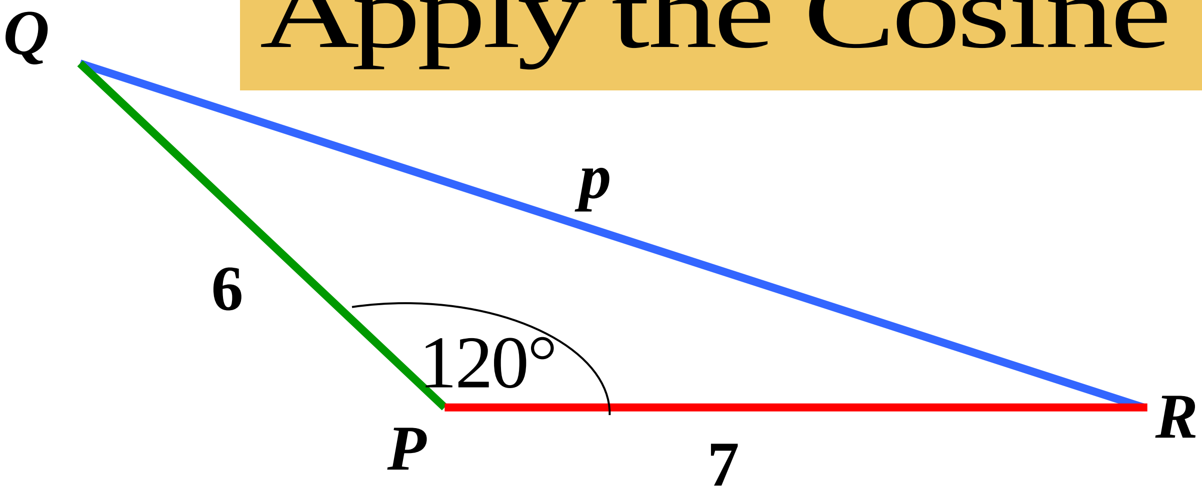


We use this form to find any angle of a triangle when we know all 3 sides.

Applications of the Cosine Rule - Example 1

Find p in the $\triangle PQR$

Apply the Cosine Rule



$$p^2 = q^2 + r^2 - 2qr\cos P$$

$$\therefore p^2 = 7^2 + 6^2 - 2(7)(6)\cos 120° = 127$$

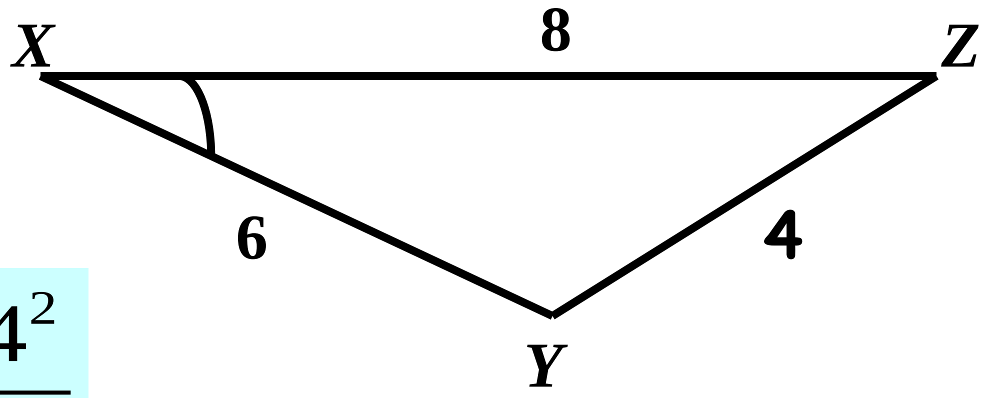
$$\therefore p = 11{,}3 \ (1 \text{ decimal accuracy})$$

Applications of the Cosine Rule - Example 2

Find $\angle X$ in the ΔXYZ

Solution: Use the Cosine Rule

$$\cos X = \frac{y^2 + z^2 - x^2}{2yz}$$

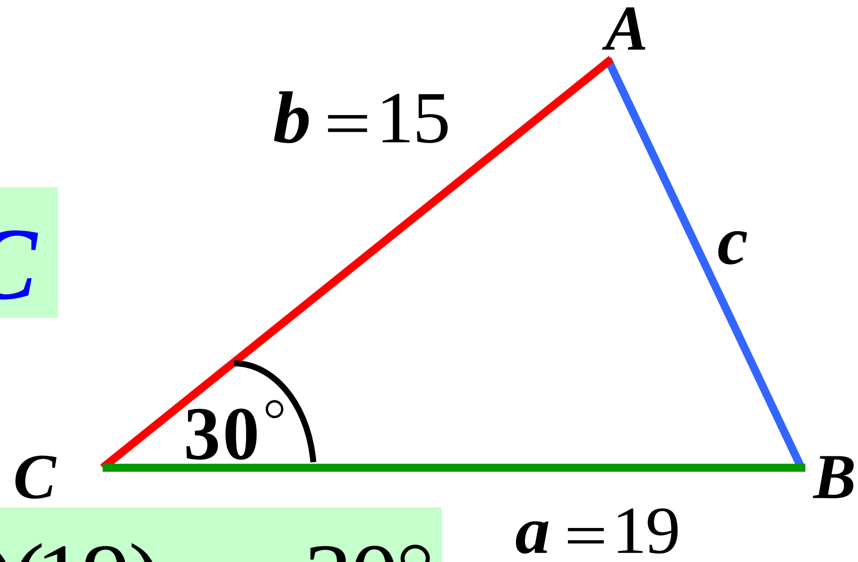


$$\Rightarrow \cos \boldsymbol{X} = \frac{8^2 + 6^2 - 4^2}{2(8)(6)}$$

$$\Rightarrow \boldsymbol{X} = 29{,}0^\circ \text{ (1 dec)}$$

Applications of the Cosine Rule – Example 3

Find side c and $\angle B$ in the given ΔABC.



Cosine rule:

$$c^2 = b^2 + a^2 - 2ba\cos C$$

$$\Rightarrow c^2 = 15^2 + 19^2 - 2(15)(19)\cos 30^\circ$$

$$\Rightarrow c = 9{,}61 \text{ (2 decimal places)}$$

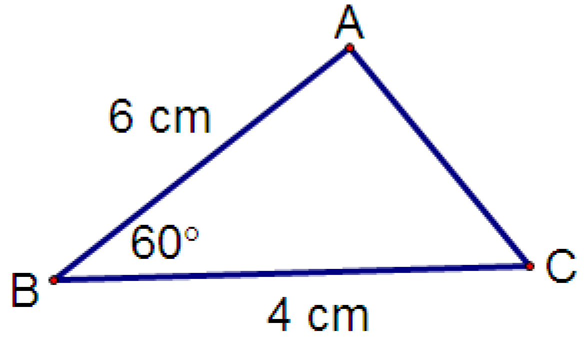
Sine rule:

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \sin B = \frac{15 \sin 30^\circ}{9{,}61}$$

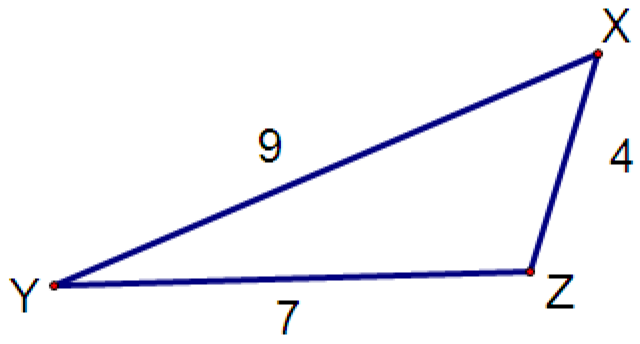
$$\Rightarrow B = 51{,}3^\circ \text{ (1 dec.)}$$

Tutorial 3: Cosine Rule

1. Given ΔABC with $AB = 6$ cm; $BC = 4$ cm and $\angle ABC = 60°$. Find AC correct to 2 decimal digits.



2. Find all the angles in ΔXYZ, giving your answers to one decimal place accuracy.



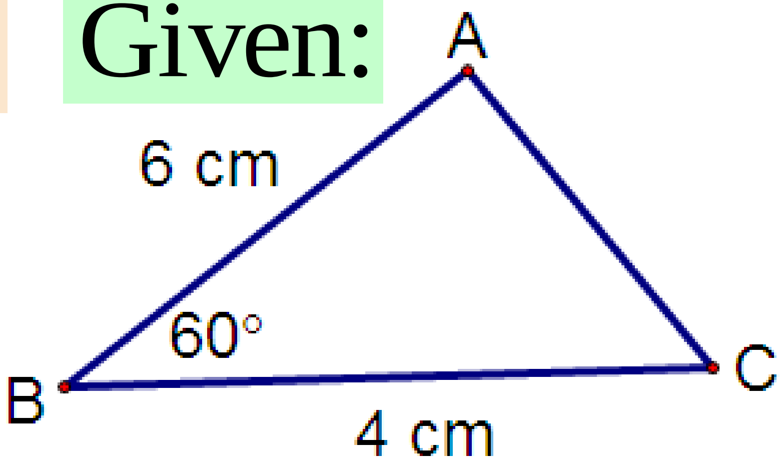
PAUSE DVD

- Do Tutorial 3
- Then View Solutions

Tutorial 3: Problem 1: Cosine Rule: Solution

Find AC (2 dec accuracy):

Given:



$$AC^2 = BC^2 + AB^2 - 2(BC)(AB)\cos\angle ABC$$

$$= 4^2 + 6^2 - 2(4)(6)\cos 60°$$

$$= 28$$

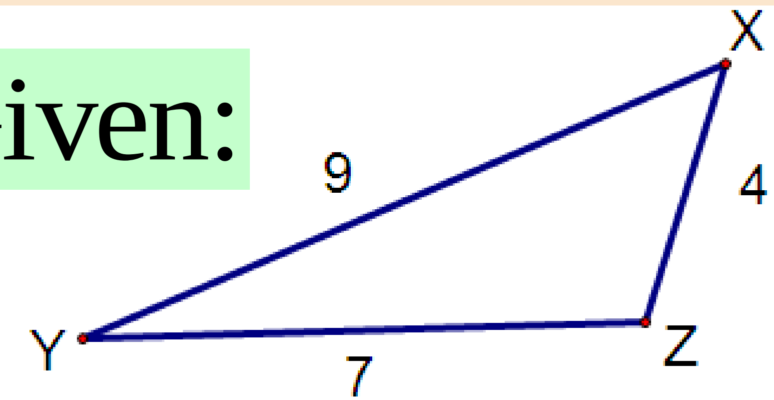
$$\therefore AC = \sqrt{28} = 5,29 \text{ cm}$$

Tutorial 3: Problem 2: Cosine Rule: Solution

Determine all angle measures of ΔXYZ.

$$\cos X = \frac{y^2 + z^2 - x^2}{2yz}$$

Given:

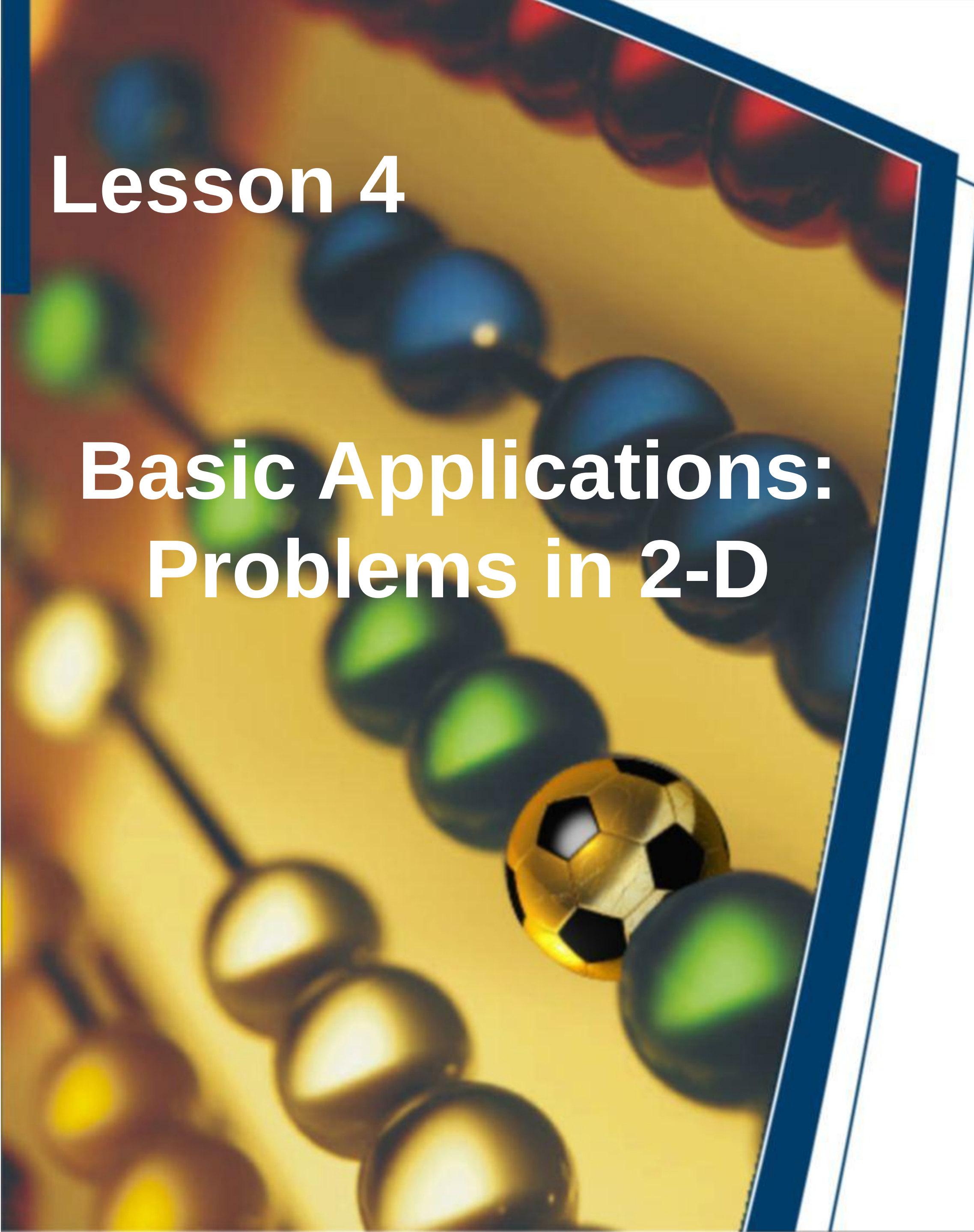


Hence $\cos X = \frac{4^2 + 9^2 - 7^2}{2(4)(9)} \Rightarrow X = 48,2°$

Now $\frac{\sin X}{x} = \frac{\sin Y}{y} \Rightarrow \sin Y = \frac{y \sin X}{x} = \frac{4 \sin 48.2°}{7}$

$\Rightarrow Y = 25,2°$

Then $Z = 180° - (X + Y) = 106,6°$



Lesson 4

Basic Applications: Problems in 2-D

Nelson Mandela Metropolitan University

for tomorrow

NCS Mathematics DVD Series

GOVAN MBEKI MATHEMATICS DEVELOPMENT UNIT

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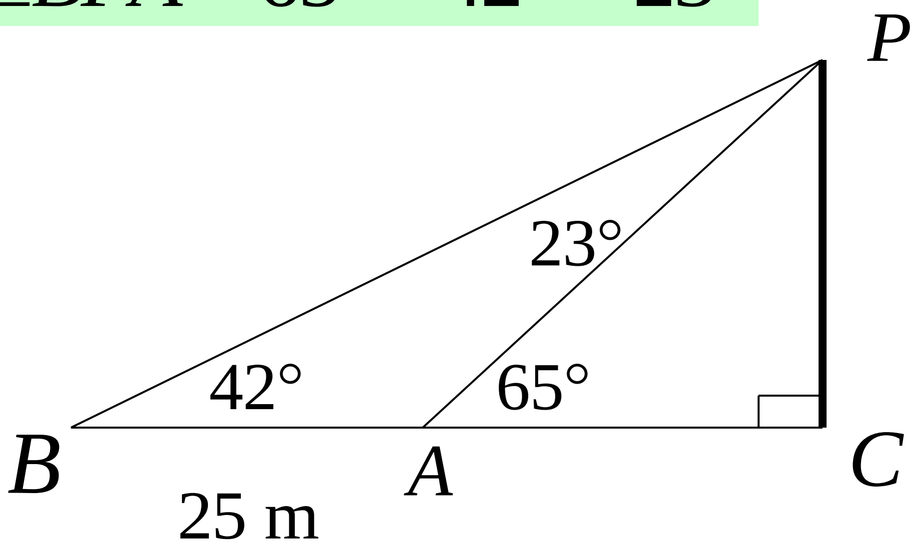
Problems in 2 – dimensions: Example 1

1. Points A and B are in the same horizontal plane as C, the foot of a vertical tower PC. $\angle B = 42°$; $\angle PAC = 65°$ and $AB = 25\ m$. Calculate PC.

Sine rule:

$$\angle BPA = 65° - 42° = 23°$$



$$\frac{AP}{\sin 42°} = \frac{25}{\sin 23°}$$

$$\therefore AP = \frac{25 \sin 42°}{\sin 23°} = 42{,}81\ \text{m}$$

$$\sin 65° = \frac{PC}{AP}$$

$$\therefore PC = AP \sin 65° = 42{,}81 \sin 65° = 38{,}8\ \text{m}$$

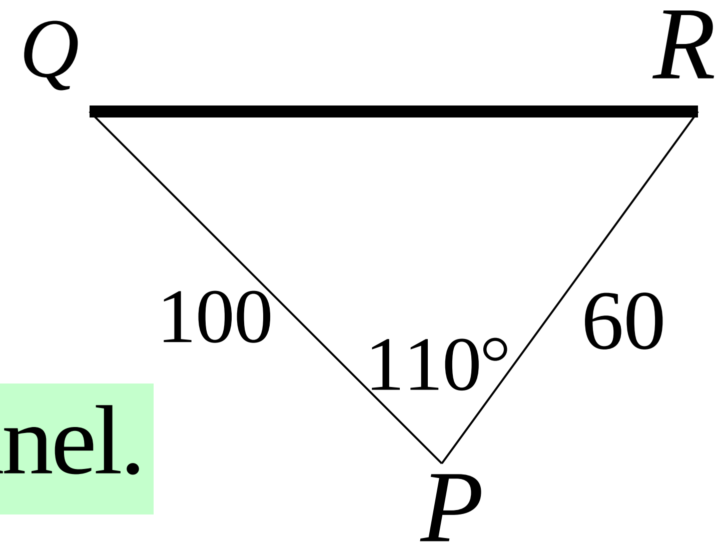
Problems in 2 – dimensions: Example 2

2. In the figure QR represents a proposed tunnel. Q and R are visible from a point P. The three points are in the same plane.

Given:

$QP = 100$ m; $PR = 60$ m

and $\angle QPR = 110°$



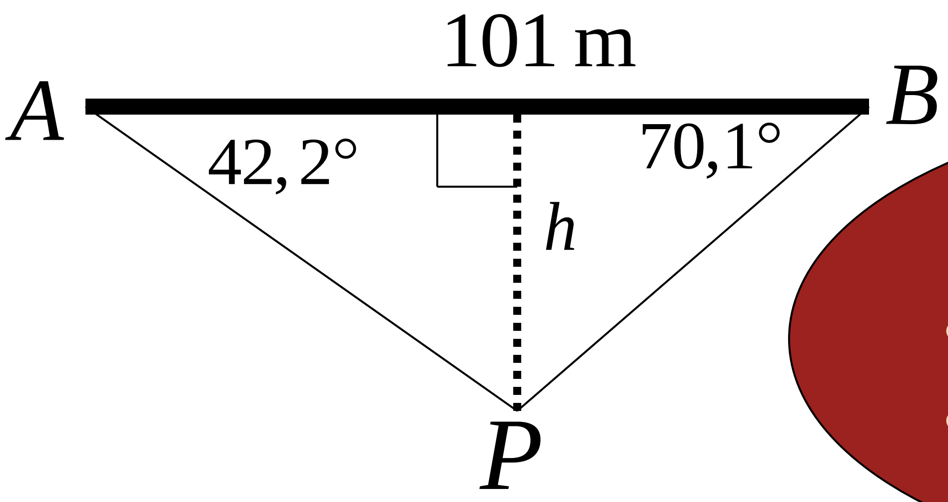
Calculate the length of tunnel.

$$QR^2 = 100^2 + 60^2 - 2(100)(60)\cos 110°$$

$$\therefore QR = 133 \text{ m}$$

Tutorial 4: Part 1: Problems in 2 – D

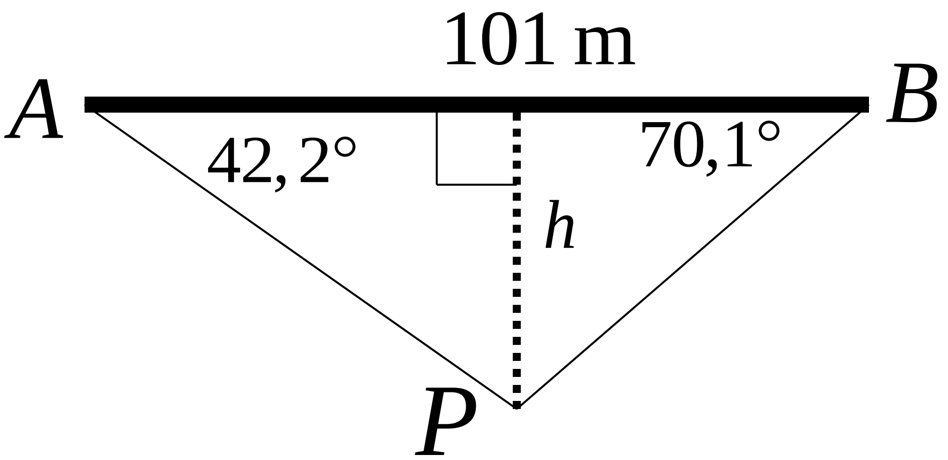
1. From the ends of a bridge AB, 101 metres long, the angles of depression of a point P on the ground directly under the bridge is 42,2° and 70,1°.
   Find the height, h, of the bridge under this point.



PAUSE DVD

- Do Tutorial 4 Part 1
- Then View Solutions

Tutorial 4: Part 1: Suggested Solution



Question: Find h

$$\angle APB = 180° - (42,2° + 70,1°) = 67,7°$$

$$\frac{AP}{\sin 70,1°} = \frac{101}{\sin 67,7°} \Rightarrow AP = \frac{101 \text{ m} \times \sin 70,1°}{\sin 67,7°}$$

$$\therefore AP = 102,65 \text{ m}$$

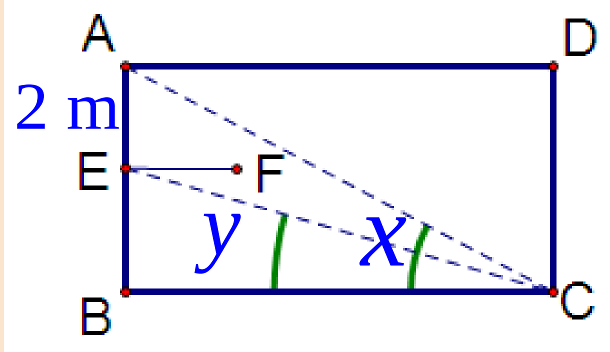
But $\sin 42,2° = \dfrac{h}{102,65}$

$$\therefore h = 102,65 \times \sin 42,2° = 68,95 \text{ m}$$

Tutorial 4: Part 2: Problems in 2-D

2. *ABCD* is a wall of a room, *AD* being the line of the ceiling. *EF* is a picture rail, with *E* being directly below *A*.

$AE = 2$ metres; $\angle ACB = x$ and $\angle ECB = y$



(a) Prove that $EC = \dfrac{2\cos x}{\sin(x-y)}$

(b) Find the length and height of the wall if $x = 33°$ and $y = 20°$

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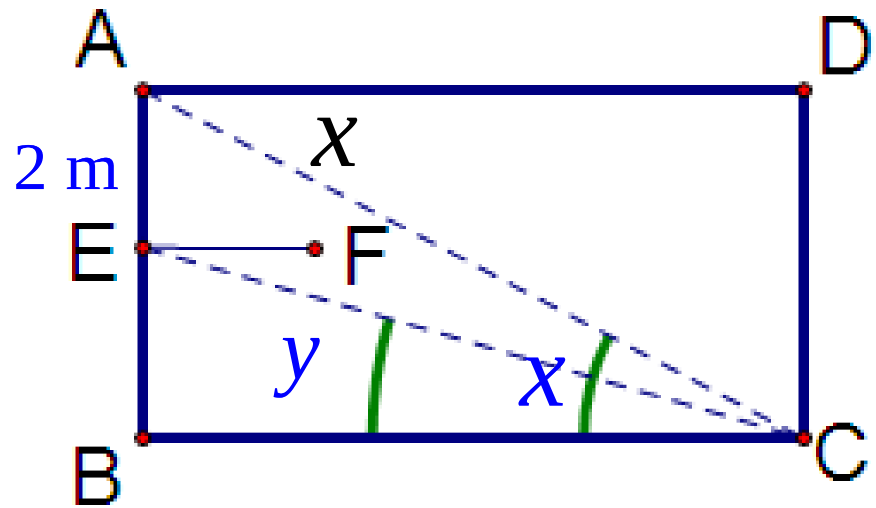
- Do Tutorial 4 Part 2
- Then View Solutions

Tutorial 4: Part 2(a) Solution

2(a) Prove that

$$EC = \frac{2\cos x}{\sin(x-y)}$$



Now $\angle ACE = x - y$ and $\angle CAD = x$

Hence, $\angle CAE = 90° - x$

From ΔAEC: $\dfrac{EC}{\sin(90° - x)} = \dfrac{2}{\sin(x-y)}$

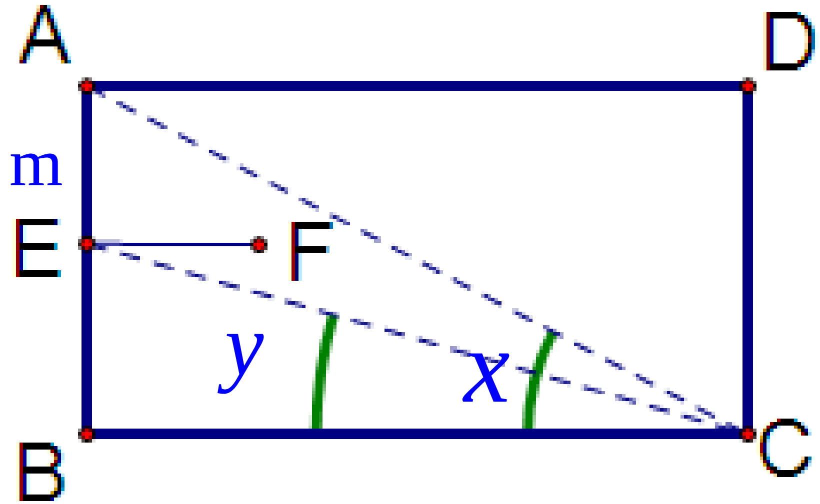
$$\therefore EC = \frac{2\sin(90° - x)}{\sin(x-y)} = \frac{2\cos x}{\sin(x-y)}$$

Tutorial 4: Part 2: Length of room

Know:

$$EC = \frac{2\cos x}{\sin(x-y)}$$

$x = 33°$ and $y = 20°$



$$EC = \frac{2\cos 33°}{\sin 13°} = 7{,}46 \text{ m}$$

$$\cos y = \frac{BC}{EC} \Rightarrow BC = EC \times \cos y$$

$$\therefore \text{ Length of room } = BC = 7{,}46 \text{ m} \times \cos 20°$$

$$= 7{,}01 \text{ m}$$

End of the DVD on Sine, Cosine and Area Rules

REMEMBER!

- Consult text-books for additional examples.
- Attempt as many as possible other similar examples on your own.
- Compare your methods with those that were discussed in the DVD.
- Repeat this procedure until you are confident.
- Do not forget:

Practice makes perfect!