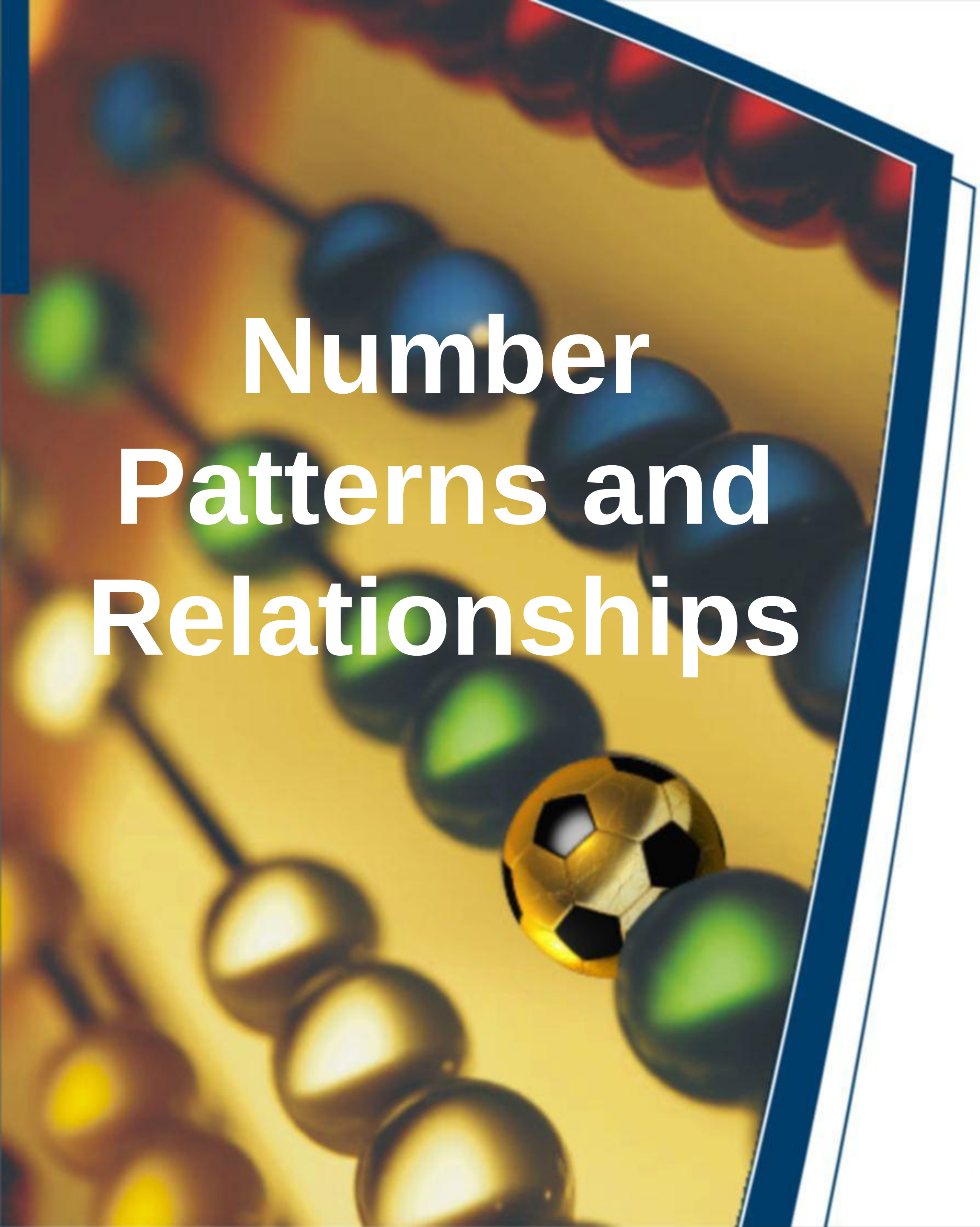

Number Patterns and Relationships

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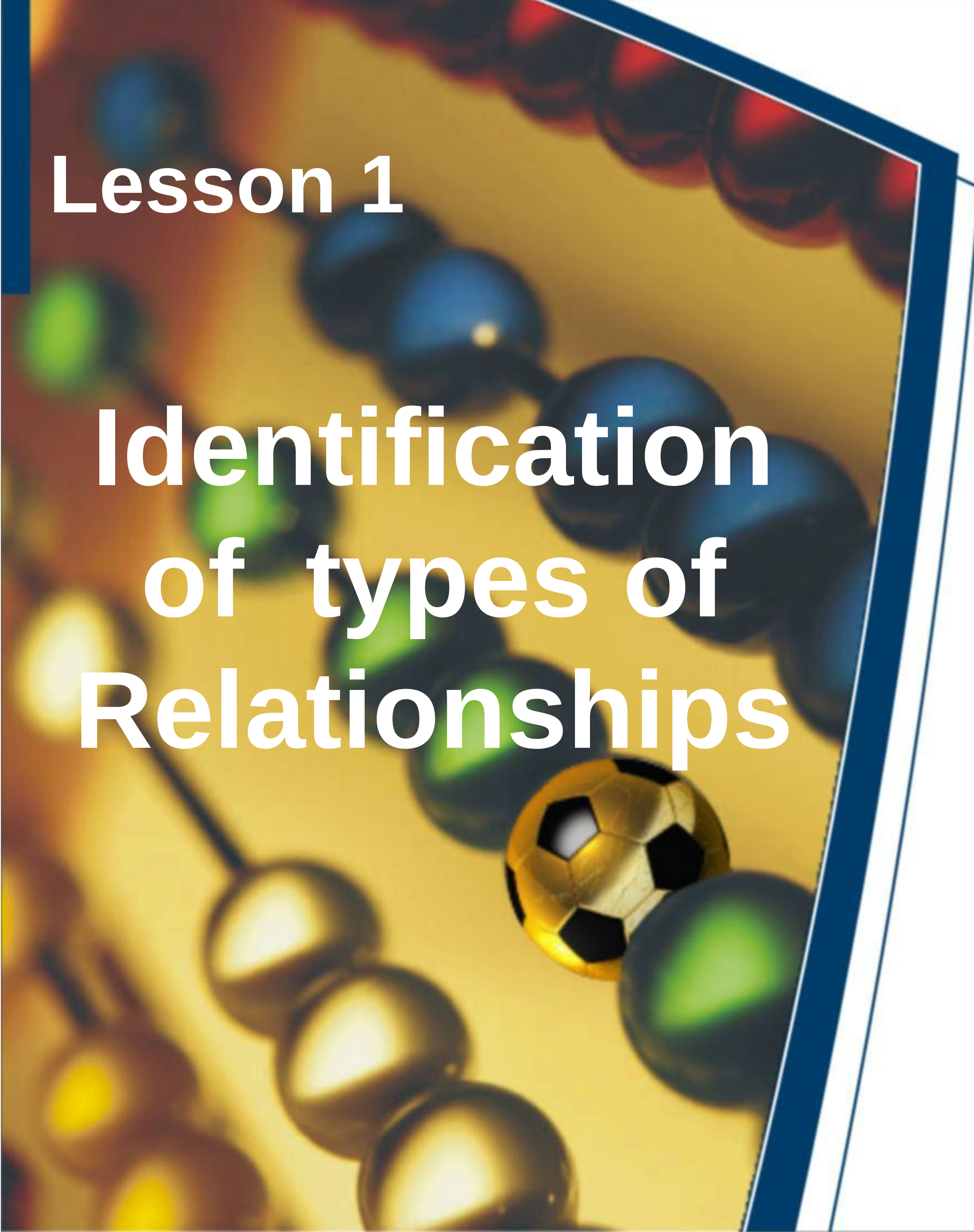
Outcomes for this DVD

In this DVD you will:

- Revise the identification of linear, quadratic and exponential relationships between two variables from graphs and from equations. LESSON 1.
- Identify types of relationships by constructing finite difference tables. LESSON 2.
- Determine the equations of relationships from finite difference tables. LESSON 3.



Lesson 1

Identification of types of Relationships

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Linear Relationship

Make Conjecture

A relationship between two variables is said to be linear if the graphical representation of the relationship results in a **straight line.**

A Linear Relationship is represented by a straight line

Example 1 Represent the following relationship graphically :

x	0	1	2	3	4
y	1	3	5	7	9

$$y = 2x + 1$$

Tutorial 1: Linear relationship in Tabular and Graphical format

Represent the relationship defined by $y = -3x + 4$

- By means of a table and
- Graphically.

PAUSE DVD

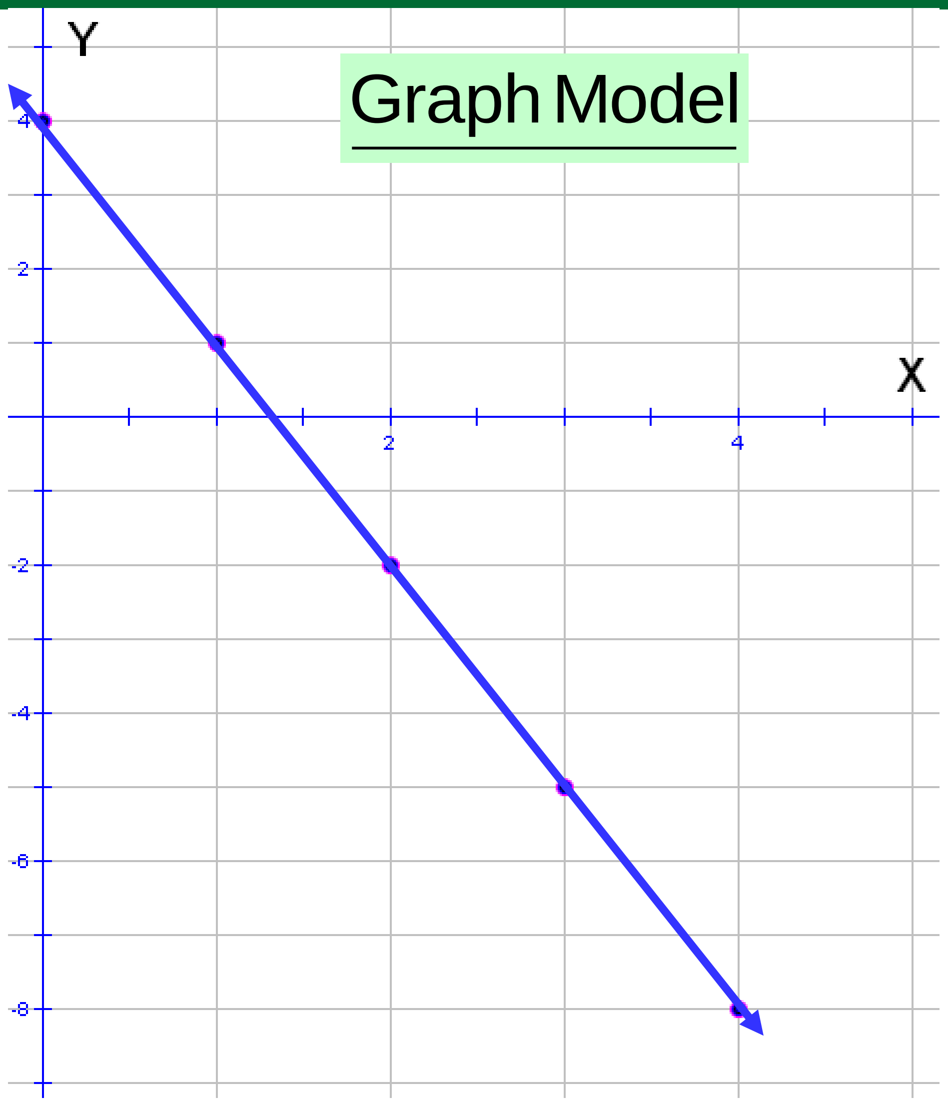
- Do Tutorial 1
- Then View Solutions

Tutorial 1: Suggested Solution

$$y = -3x + 4$$

Table Model

x	0	1	2	3	4
y	4	1	-2	-5	-8



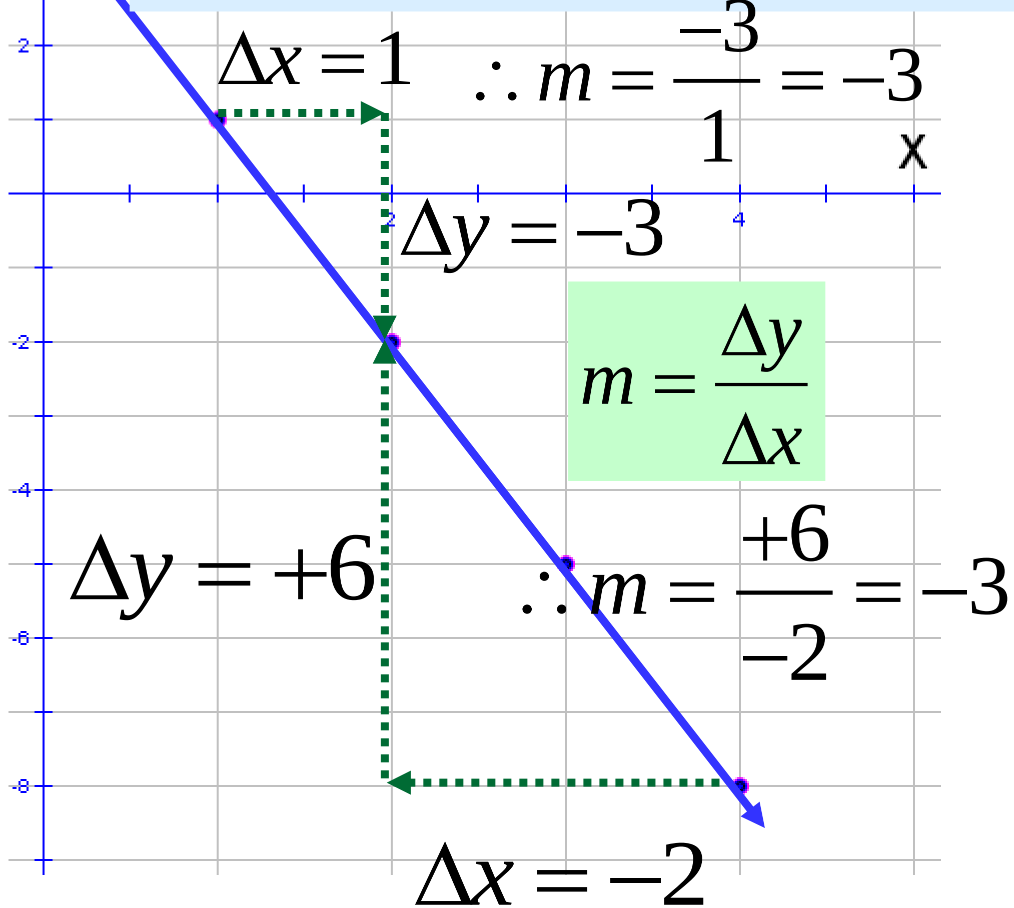
Important Notes on Linear Relationships

$$y = -3x + 4$$

$*$ The **standard form** of the equation of a linear relationship is: $y = mx + c$

$*$ m is the measure of the **gradient** of the line.

$*$ $(0\,;c)$ is the point where the line **intersects the y - axis.**

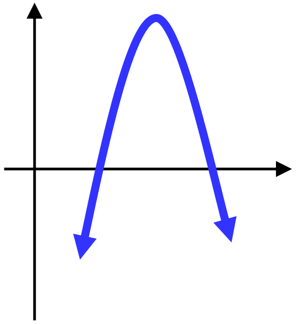


Quadratic Relationship

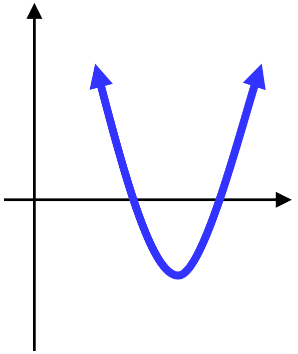
Make Conjecture

A relationship between two variables is said to be quadratic if the graphical representation of the relationship results in a parabola.

Recall that the shape of a parabola can either be:



OR



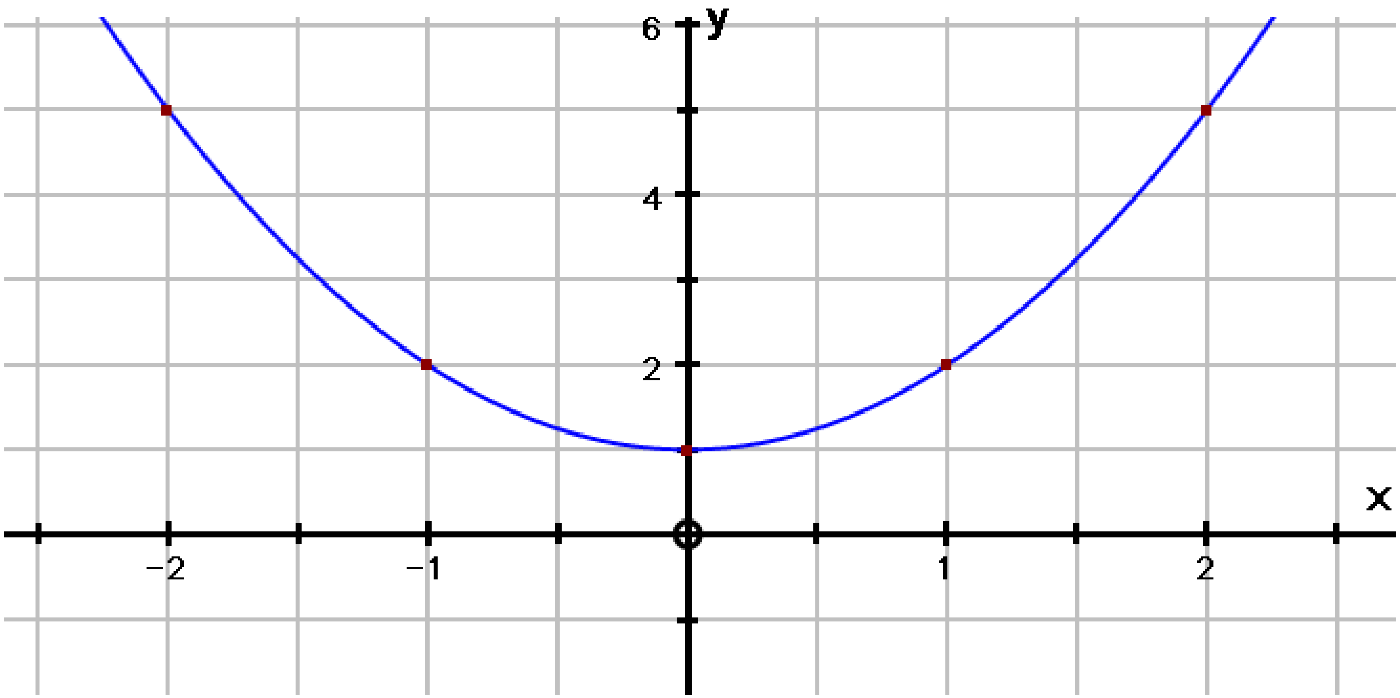
A Quadratic Relationship is represented by a parabola

Example 2

Represent the following relationship graphically :

x	–2	–1	0	1	2
y	5	2	1	2	5

$$y = x^2 + 1$$



Tutorial 2: Quadratic relationship in Tabular and Graphical format

Represent the relationship defined by $y = -x^2 + 2x + 3$

- By means of a table and
- Graphically.

PAUSE DVD

- Do Tutorial 2
- Then View solutions

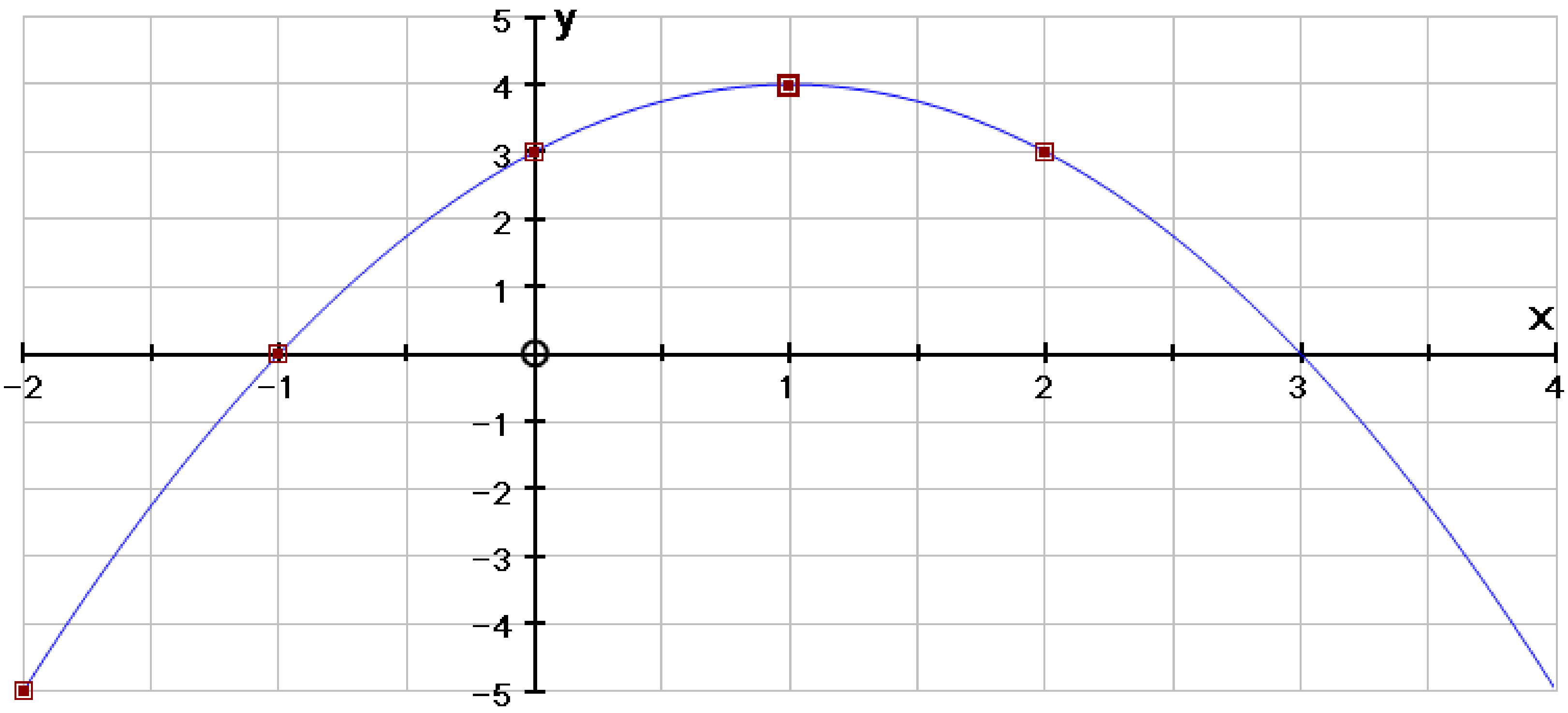
Tutorial 2: Suggested Solutions

$$y = -x^2 + 2x + 3$$

Table Model

x	-2	-1	0	1	2
y	-5	0	3	4	3

Graph Model



Important Notes on Quadratic Relationships

∗ The **standard form** of the equation of a quadratic relationship is:

$y = ax^2 + bx + c$ *where* $a \neq o$.

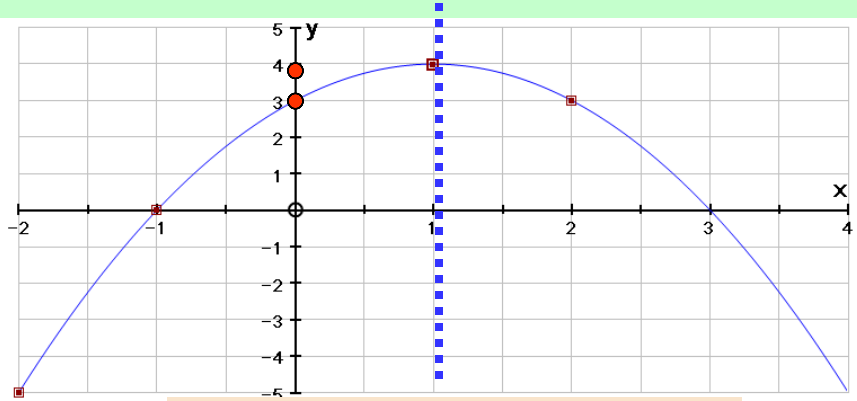
∗ Max value given by:

$$y = -\frac{\Delta}{4a} = -\frac{16}{-4} = 4$$

∗ $(0;c)$ is the point where the parabola **intersects the y - axis.**

∗ The **line of symmetry** is given by:

$$x = -\frac{b}{2a} = -\frac{2}{-2} = 1$$

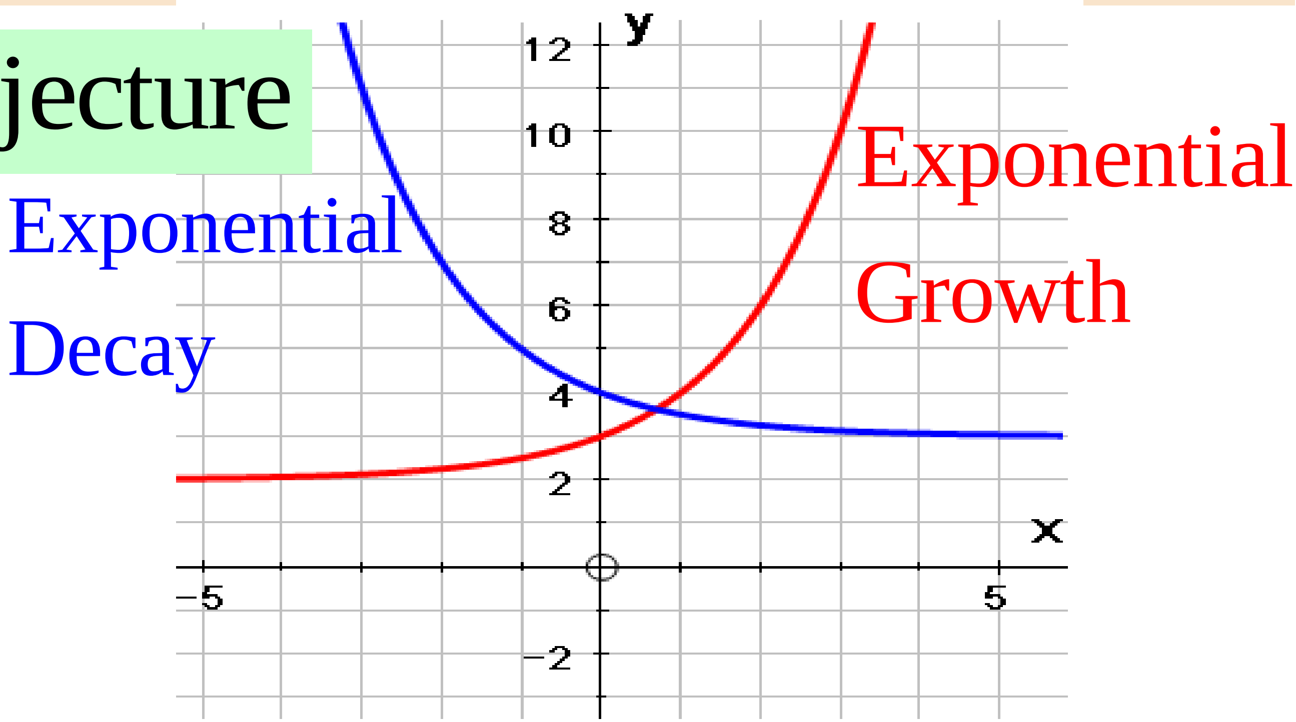


$y = -x^2 + 2x + 3$

The Exponential Relationship

A relationship between two variables is said to be exponential if the graphical representation results in a curve which is similar to one of the two graphs shown below.

Make Conjecture



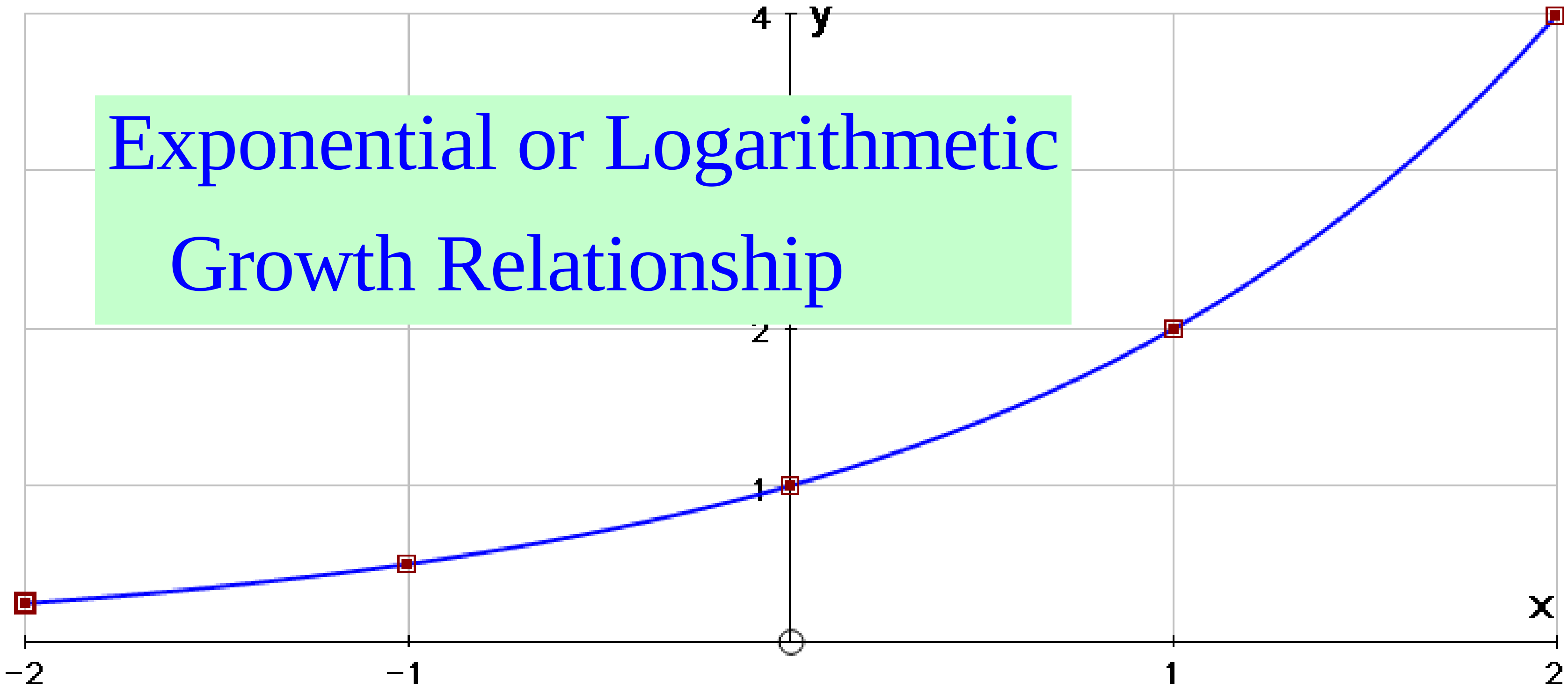
Representation of an Exponential Relationship

Example 3

Represent the following relationship graphically :

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$$y = 2^x \Leftrightarrow \log_2 y = x$$



Tutorial 3: Exponential relationship in Tabular and Graphical format

Represent the relationship defined

by $y = 3^{-x}$

- By means of a table and
- Graphically.

PAUSE DVD

- Do Tutorial 3
- Then View Solutions

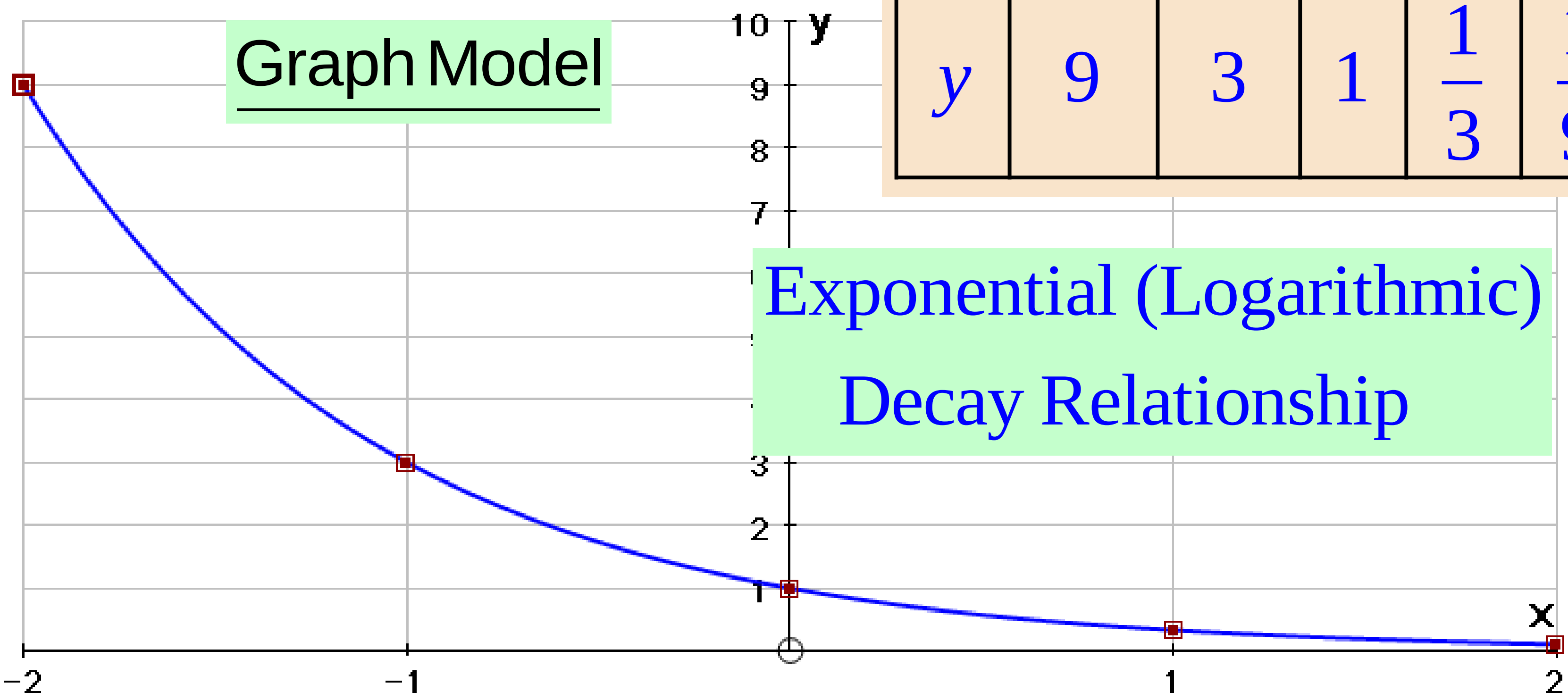
Tutorial 3: Suggested Solution

$$y = 3^{-x} = \left(\frac{1}{3}\right)^x \Leftrightarrow \log_{\frac{1}{3}} y = x$$

Table Model

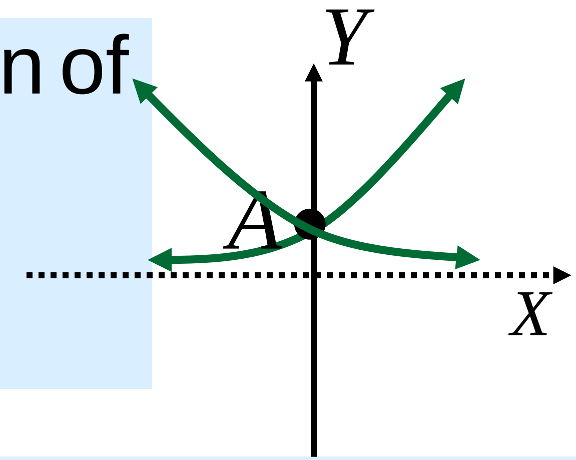
x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

Graph Model



Exponential (Logarithmic) Decay Relationship

Important Notes on Exponential Relationships



* The **standard form** of the equation of an exponential relationship is:

  $y = Ab^x$ *where* $b > 0, b \neq 1.$

* The line of $y = 0$ ($x-$axis) is the horizontal asymptote.

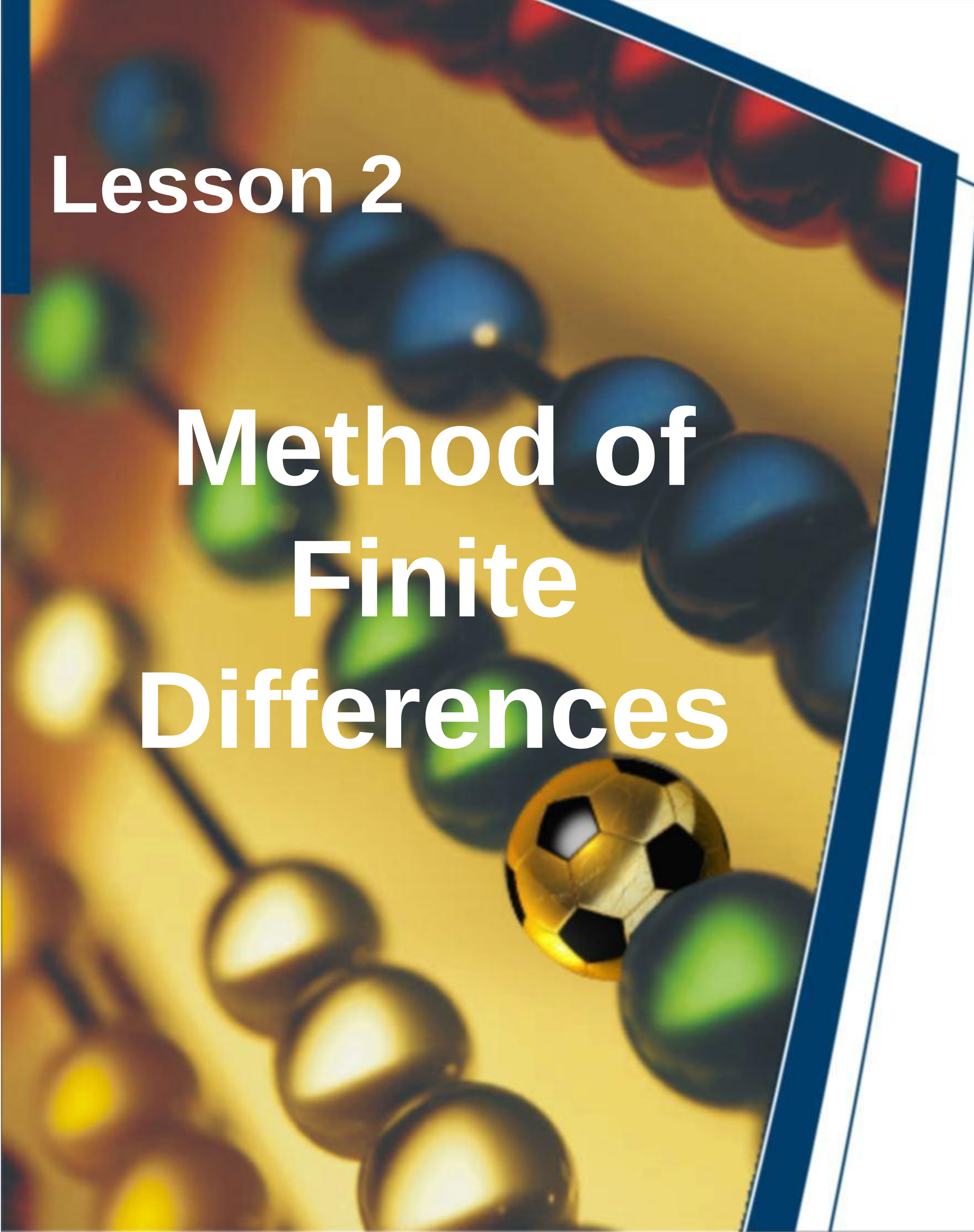
* $(0; A)$ is the point where this exponential relation intersects the $y-$axis.

Finding a more precise method

* The type of relationship can thus be identified from their respective graph or equation.
* Graphing the relationship could be time consuming and confusing.
* Moreover, we may not be given the equation of the relationship.
* Thus, we now introduce an alternative method for determining whether a relationship is linear, quadratic or exponential.

Lesson 2

Method of Finite Differences



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Constant First Differences Imply a Linear Relationship

Example 4

Let us consider the following relationship that is given in tabular form:

$$y = 3x + 4$$

x	0	1	2	3	4
y	4	7	10	13	16
First Differences	$7-4=3$	$10-7=3$	$13-10=3$	$16-13=3$	

- In this example, the first differences are constant.
- In such a case we can conclude that the relationship is linear (Check graphically).

Constant Second Differences Imply a Quadratic Relationship

Example 5

Let us consider the following relationship that is given in tabular form: $y = 3x^2 + 2x + 1$

x	0	1	2	3	4
y	1	6	17	34	57
First Differences	$6-1=5$	$17-6=11$	$34-17=17$	$57-34=23$	
Second Differences	$11-5=6$	$17-11=6$	$23-17=6$		

- In this example, the first differences are not constant but second differences turn out to be constant.
- In such a case we can conclude that the relationship is quadratic (Check graphically).

Constant Ratios
Imply an Exponential Relationship

Example 6

Let us consider the following relationship that is given in tabular form:

x	0	1	2	3	4
y	1	2	4	8	16
First Differences	$2-1=1$	$4-2=2$	$8-4=4$	$16-8=8$	
Second Differences	$2-1=1$	$4-2=2$	$8-4=4$		

- In this example, neither the first differences nor the second differences are constant.
- In such a case we investigate the ratio between consecutive values in the last three rows.

Ratios in last three rows

x	0	1	2	3	4
y	1	2	4	8	16
First Differences	$2-1=1$	$4-2=2$	$8-4=4$	$16-8=8$	
Second Differences	$2-1=1$	$4-2=2$	$8-4=4$		

y – row (Output): $\frac{16}{8}=\frac{8}{4}=\frac{4}{2}=\frac{2}{1}=2$

$$y=2^x \Leftrightarrow \log_2 y = x$$

1^{st} Difference row: $\frac{8}{4}=\frac{4}{2}=\frac{2}{1}=2$

2^{nd} Difference row: $\frac{4}{2}=\frac{2}{1}=2$

- Note in all three cases the ratios are constant and equal to two.

- Such constant ratios imply that the relationship is exponential or logarithmic (Check graphically).

Tutorial 4: Identifying types of relationships

Use the method of finite differences to determine the type of relationship shown in the following tables:

(1)

x	0	1	2	3
y	6	12	20	30

(2)

x	0	1	2	3
y	4	7	10	13

(3)

p	0	1	2	3
q	1	3	9	27

(4)

r	10	11	12	13
s	12	19	26	33

(5)

x	7	8	9	10
y	9	16	29	48

(6)

u	2	3	4	5
v	16	32	64	128

PAUSE DVD

- Do Tutorial 4
- Then View Solutions

Tutorial 4: Suggested Solutions to Problems 1 and 2

(1)

x	0	1	2	3
y	6	12	20	30
1^{st}	6	8	10	
2^{nd}	2	2		

$\therefore$ Second Differences are constant

$\Rightarrow$ Relationship is quadratic

(2)

x	0	1	2	3
y	4	7	10	13
1^{st}	3	3	3	

$\therefore$ First Differences are constant

$\Rightarrow$ Relationship is linear

Tutorial 4:
Suggested Solution to Problem 3

(3)

p	0	1	2	3
q	1	3	9	27
1^{st}	2	6	18	
2^{nd}	4	12		
Ratio Row 2	$\frac{3}{1}=3$	$\frac{9}{3}=3$	$\frac{27}{9}=3$	
Ratio Row 3	$\frac{6}{2}=3$	$\frac{18}{6}=3$		
Ratio Row 4	$\frac{12}{4}=3$	More info?		

$\therefore$ Relationship is exponential because ratios are constant.

Tutorial 4:
Suggested Solutions to Problems 4 and 5

(4)

r	10	11	12	13
s	12	19	26	33
1^{st}	7	7	7	

$\therefore$ Relationship is linear because first differences are constant.

(5)

x	7	8	9	10
y	9	16	29	48
1^{st}	7	13	19	
2^{nd}	6	6		

$\therefore$ Relationship is quadratic because second differences are constant.

Tutorial 4:
Suggested Solution to Problem 6

(6)

u	2	3	4	5
v	16	32	64	128
1^{st}	16	32	64	
2^{nd}	16	32		
Ratio Row 3	$\frac{32}{16} = 2$	$\frac{64}{32} = 2$	$\frac{128}{64} = 2$	
Ratio Row 4	$\frac{32}{16} = 2$	$\frac{64}{32} = 2$		
Ratio Row 5	$\frac{32}{16} = 2$	**Enough Info?**		

$\therefore$ Relationship is exponential because ratios are constant.

Lesson 3

Equations from Finite Differences

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Finding Equations of Linear Relationships

A linear relationship is defined by : $y = mx + c$.

x	y	First Difference
0	c	$(m+c)-c=m$
1	$m+c$	$(2m+c)-(m+c)=m$
2	$2m+c$	$(3m+c)-(2m+c)=m$
3	$3m+c$	$(4m+c)-(3m+c)=m$
4	$4m+c$	

- First differences are constant and coincides with the gradient of the line, m. (**Warning!**)
- Furthermore, c is the value of y when $x = 0$.

Warning related to value of m!

A linear relationship is defined by : $y = mx + c$.

x	y	First Difference
0	c	$(m+c)-c=m$
1	$m+c$	$(2m+c)-(m+c)=m$
2	$2m+c$	$(3m+c)-(2m+c)=m$
3	$3m+c$	$(4m+c)-(3m+c)=m$
4	$4m+c$	

$$\Delta x = 1 \Rightarrow m = \text{ First Difference.}$$

What if $\Delta x = 2$?

x	y	First Difference
0	c	$(2m+c)-c=2m$
2	$2m+c$	$(4m+c)-(3m+c)=2m$
4	$4m+c$	$(6m+c)-(4m+c)=2m$
6	$6m+c$	$(8m+c)-(6m+c)=2m$
8	$8m+c$	

x	y	First Difference
0	c	$(3m+c)-c=3m$
3	$3m+c$	$(6m+c)-(3m+c)=3m$
6	$6m+c$	$(9m+c)-(6m+c)=3m$
9	$9m+c$	$(12m+c)-(9m+c)=3m$
12	$12m+c$	

$$\therefore \Delta x = 2 \Rightarrow m = \frac{\text{First Difference}}{2}$$

$$m = \frac{\text{First Difference}}{3}$$

Find the equation of a Linear Relationship

Example 7 Find the equation of the following relationship :

x	0	1	2	3
y	7	11	15	19

Solution

x	0	1	2	3
y	7	11	15	19
1^{st}	4	4	4	

Check!

- First differences are constant and has a value of 4 $\Rightarrow m = 4.$
- $y = 7$ when $x = 0 \Rightarrow c = 7.$

$\therefore y = 4x + 7$ is the defining equation of this linear relationship.

Finding Equations of Quadratic Relationships

A quadratic relationship is defined by : $y = ax^2 + bx + c; a \neq 0$

x	y	First Difference	Second Differences
0	c	$a+b$	$2a$
1	$a+b+c$	$3a+b$	$2a$
2	$4a+2b+c$	$5a+b$	$2a$
3	$9a+3b+c$	$7a+b$	**Warning!**
4	$16a+4b+c$		

- Second differences are constant and is twice the coefficient of x^2. So we can find the value of a.
- Furthermore, c is the value of y when $x = 0$.
- To find the value of b, we can substitute any (x, y) − pair.

Find the equation of a Quadratic Relationship

Example 8

Find the equation of the following relationship:

x	0	1	2	3
y	5	6	13	26

Solution

x	0	1	2	3
y	5	6	13	26
1^{st}	1	7	13	
2^{nd}	6	6		

Check!

- Second differences are constant with a value of 6 $\Rightarrow 2a = 6 \Rightarrow a = 3.$
- $y = 5$ when $x = 0 \Rightarrow c = 5.$

$\Rightarrow y = 3x^2 + bx + 5$ is the defining equation at this stage.

- Substitute $(1;6)$ into $y = 3x^2 + bx + 5 \Rightarrow 6 = 3 + b + 5 \Rightarrow b = -2.$

$\therefore\ y = 3x^2 - 2x + 5$ defines this quadratic relationship.

Finding Equations of Exponential Relationships

An exponential relationship is defined by : $y = Ab^x$ where $b > 0,\ b \neq 1.$

x	y	First Difference	Second Differences
0	A	$A(b-1)$	$A(b-1)(b-1)$
1	Ab	$Ab(b-1)$	$Ab(b-1)(b-1)$
2	Ab^2	$Ab^2(b-1)$	$Ab^2(b-1)(b-1)$
3	Ab^3	$Ab^3(b-1)$	
4	Ab^4		

$$\frac{Ab}{A} = \frac{Ab^2}{Ab} = \frac{Ab^3}{Ab^2} = \frac{Ab^4}{Ab^3} = b$$

- Ratios in colums 2, 3 and 4 are constant. (Check!)
- So we have the value of b.

Warning!

- The value of A is the value of y when $x = 0$.

Find the equation of an Exponential Relationship

Example 9

Find the equation of the following relationship:

x	0	1	2	3
y	5	10	20	40

Solution

Only need to check output ratios!

x	0	1	2	3
y	5	10	20	40
1^{st}	5	10	20	
2^{nd}	5	10		
Ratio in Row 2	$\frac{10}{5}=2$	$\frac{20}{10}=2$	$\frac{40}{20}=2$	
Ratio in Row 3	$\frac{10}{5}=2$	$\frac{20}{10}=2$		
Ratio in Row 4	$\frac{10}{5}=2$			

- Constant Ratio of 2 $\Rightarrow b=2.$
- $y=5$ when $x=0 \Rightarrow A=5.$

$\therefore\ y=5\times 2^x$ defines this exponential relationship.

Check!

Tutorial 5: Finding the Defining Equation of a Relationship

Determine the defining equation for each relationship shown in the following tables:

(1)

x	0	1	2	3
y	6	12	20	30

(2)

x	0	1	2	3
y	4	7	10	13

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- Do Tutorial 5
- Then View Solutions

(3)

p	0	1	2	3
q	1	3	9	27

(4)

r	10	11	12	13
s	12	19	26	33

Tutorial 5:
Suggested Solution to Problem 1

(1)

x	0	1	2	3
y	6	12	20	30
1^{st}	6	8	10	
2^{nd}	2	2		

$\therefore$ Second Differences are constant

$\Rightarrow$ Relationship is quadratic

- $y = ax^2 + bx + c$ is the defining equation.
- $2a = 2 \Rightarrow a = 1$
- $y = 6$ when $x = 0 \Rightarrow c = 6$

$\therefore$ At this stage the equation is $y = x^2 + bx + 6$.

Substitute $(1;12)$ into $y = x^2 + bx + 6$:

$\Rightarrow 12 = 1 + b + 6 \Rightarrow b = 5$

Check!

$\therefore$ $y = x^2 + x + 6$ is the defining quadratic equation.

Tutorial 5:
Suggested Solution to Problem 2

(2)

x	0	1	2	3
y	4	7	10	13
1^{st}	3	3	3	

$\therefore$ First Differences are constant

$\Rightarrow$ Relationship is linear

- $y = mx + c$ is the defining equation.
- $m = 3$
- $y = 4$ when $x = 0$

$\Rightarrow c = 4$

$\therefore$ $y = 3x + 4$ defines this linear relationship.

Check!

Tutorial 5: Suggested Solution to Problem 3

(3)

p	0	1	2	3
q	1	3	9	27
1^{st}	2	6	18	
2^{nd}	4	12		
Output Ratio (Ratio Row 2)	$\frac{3}{1}=3$	$\frac{9}{3}=3$	$\frac{27}{9}=3$	

$\therefore$ Output Ratios constant $\Rightarrow$ Relationship is exponential

- $q = A \times b^p$ defines this relationship.

- Constant ratio is 3

$\therefore\ b = 3$

- $q = 1$ when $p = 0$

$\therefore 1 = A \times 3^0$

$\therefore\ A = 1$

$\therefore\ q = 3^p$ is the defining equation for this exponential relationship.

Check!

Tutorial 5:
Suggested Solution to Problem 4

(4)

r	10	11	12	13
s	12	19	26	33
1^{st}	7	7	7	

$\therefore$ Relationship is linear because first differences are constant.

- $s = mr + c$ is the defining equation.

- $m = 7$

$\therefore \ s = 7r + c$ is the defining equation.

Substitute $(10;12)$ into $s = 7r + c$:

$\Rightarrow 12 = 7 \times 10 + c \Rightarrow c = -58$

Check!

$\therefore \ s = 7r - 58$ defines this linear relationship.

End of the DVD on Number Patterns and Relationships

REMEMBER!

- Consult text-books for additional examples.
- Attempt as many as possible other similar examples on your own.
- Compare your methods with those that were discussed in the DVD.
- Repeat this procedure until you are confident.
- Do not forget:

Practice makes perfect!