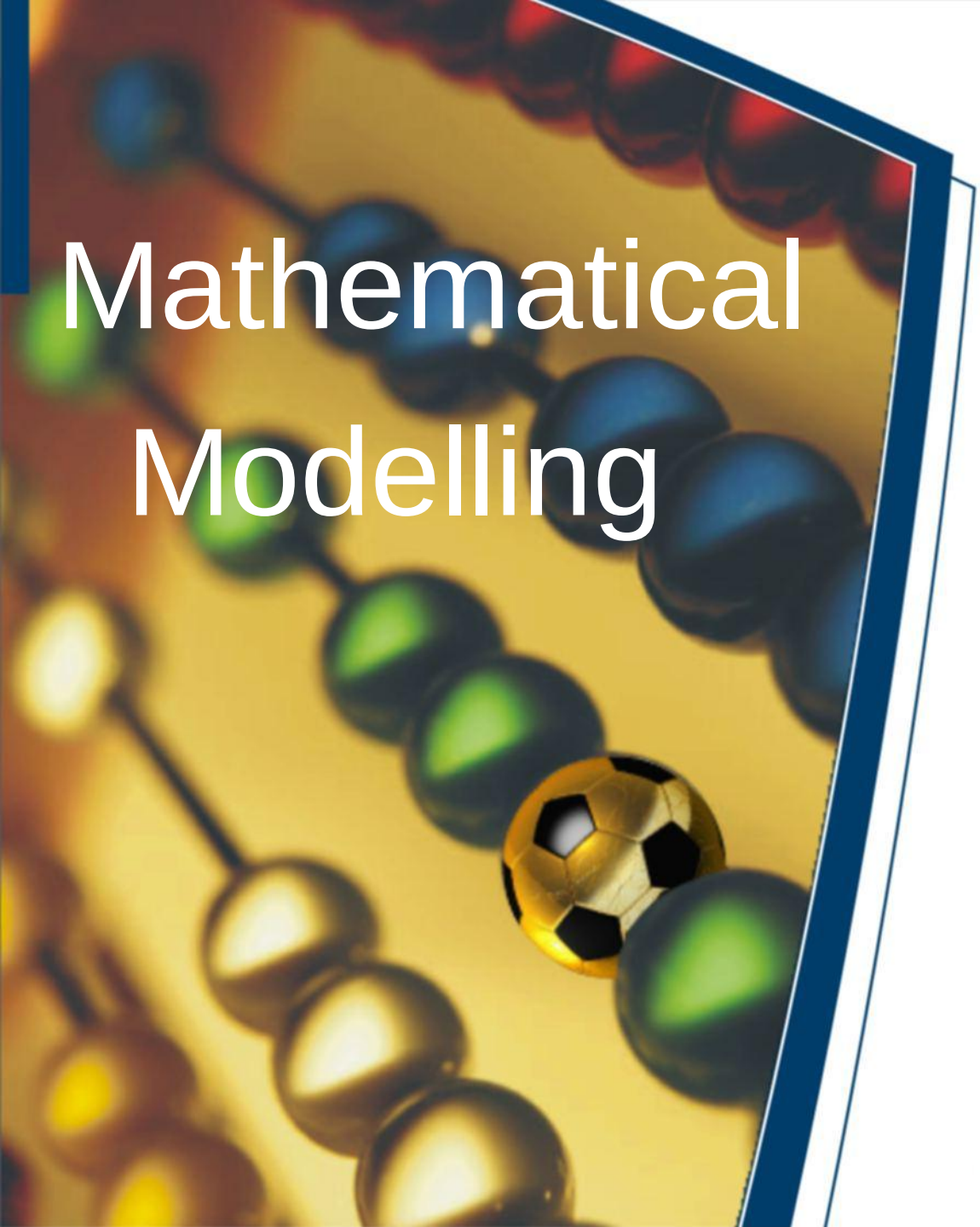

Mathematical Modelling

Nelson Mandela Metropolitan University

for tomorrow

NCS Mathematics DVD Series

GOVAN MBEKI MATHEMATICS DEVELOPMENT UNIT

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Outcomes for this DVD

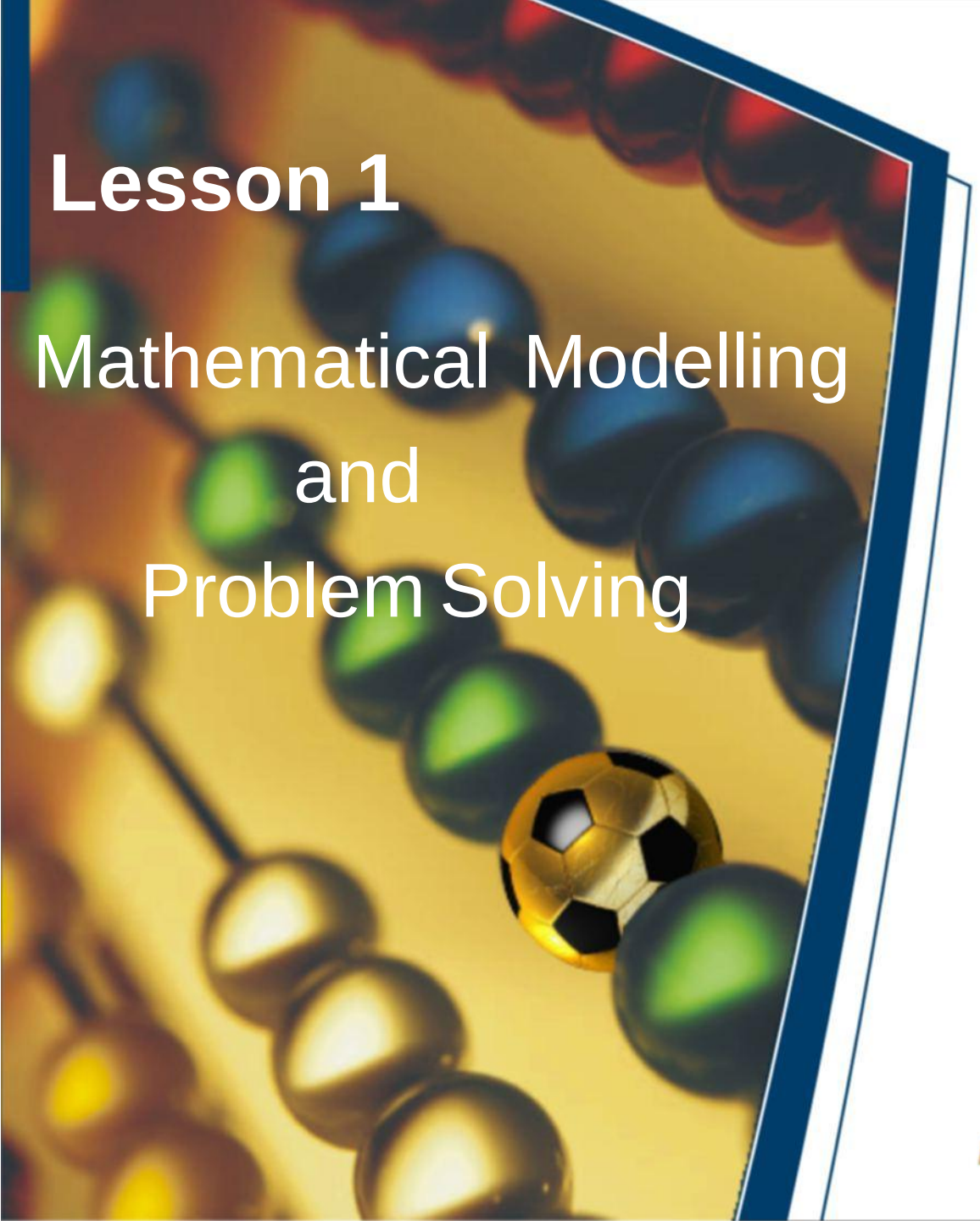
In this DVD we will:

- Discuss the relationship between Mathematical Modelling and Problem Solving.

LESSON 1

- Apply Mathematical Modelling and Problem Solving in different contexts.

LESSON 2



Lesson 1

Mathematical Modelling and Problem Solving

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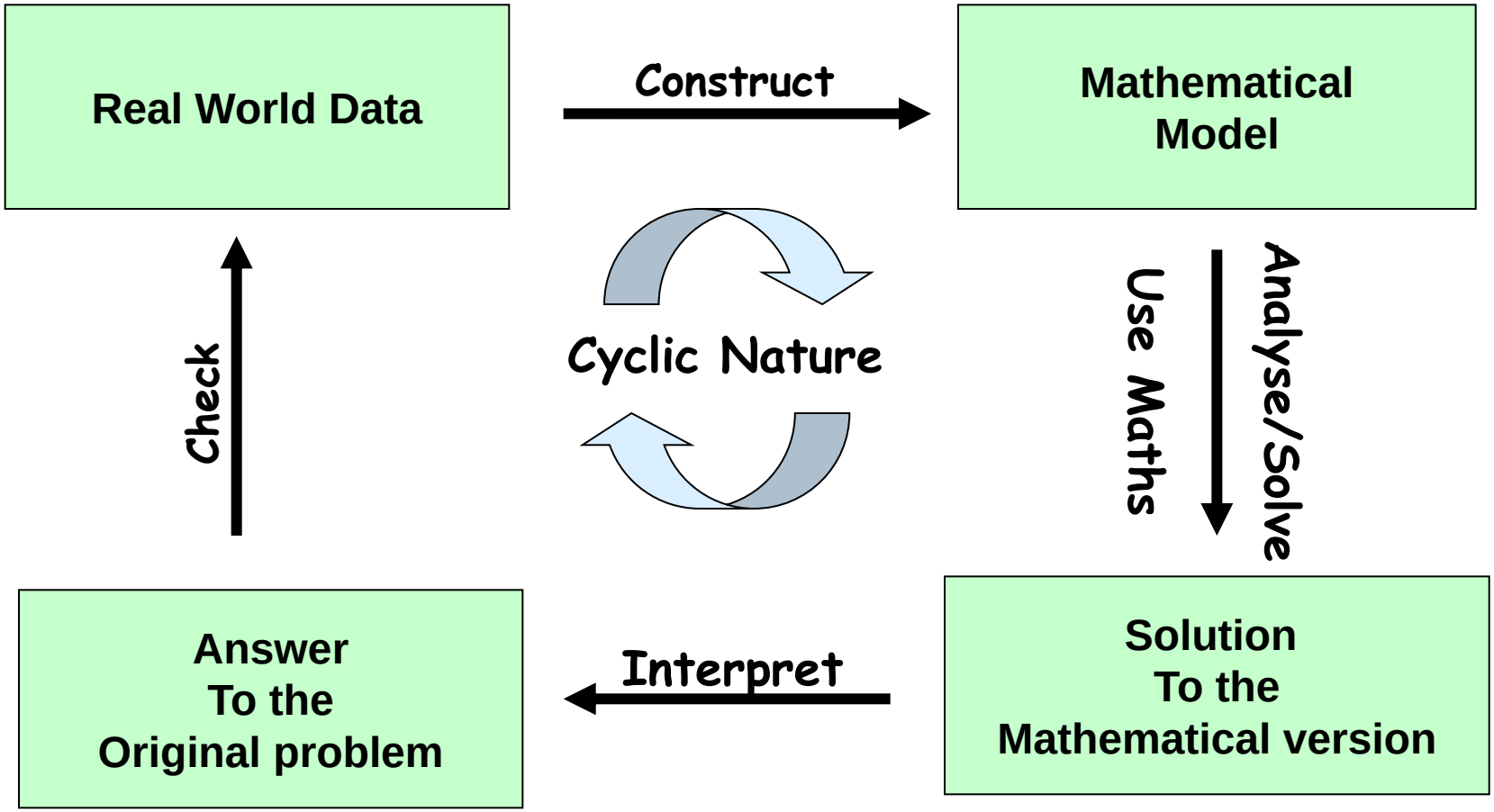
What is Mathematical Modelling?

- The process of representing real-world situations through Mathematics is called Mathematical Modelling.
- Modelling process can be represented as follows:



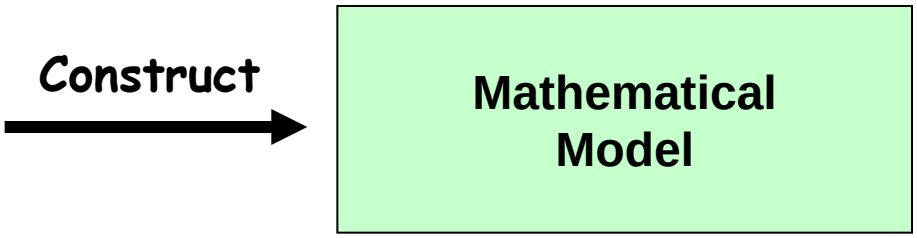
Which representations can be used in Mathematical Modelling?

- Graphical or Pictorial representations
- Symbolic representations
- Tabular or numerical representations
- Verbal representations

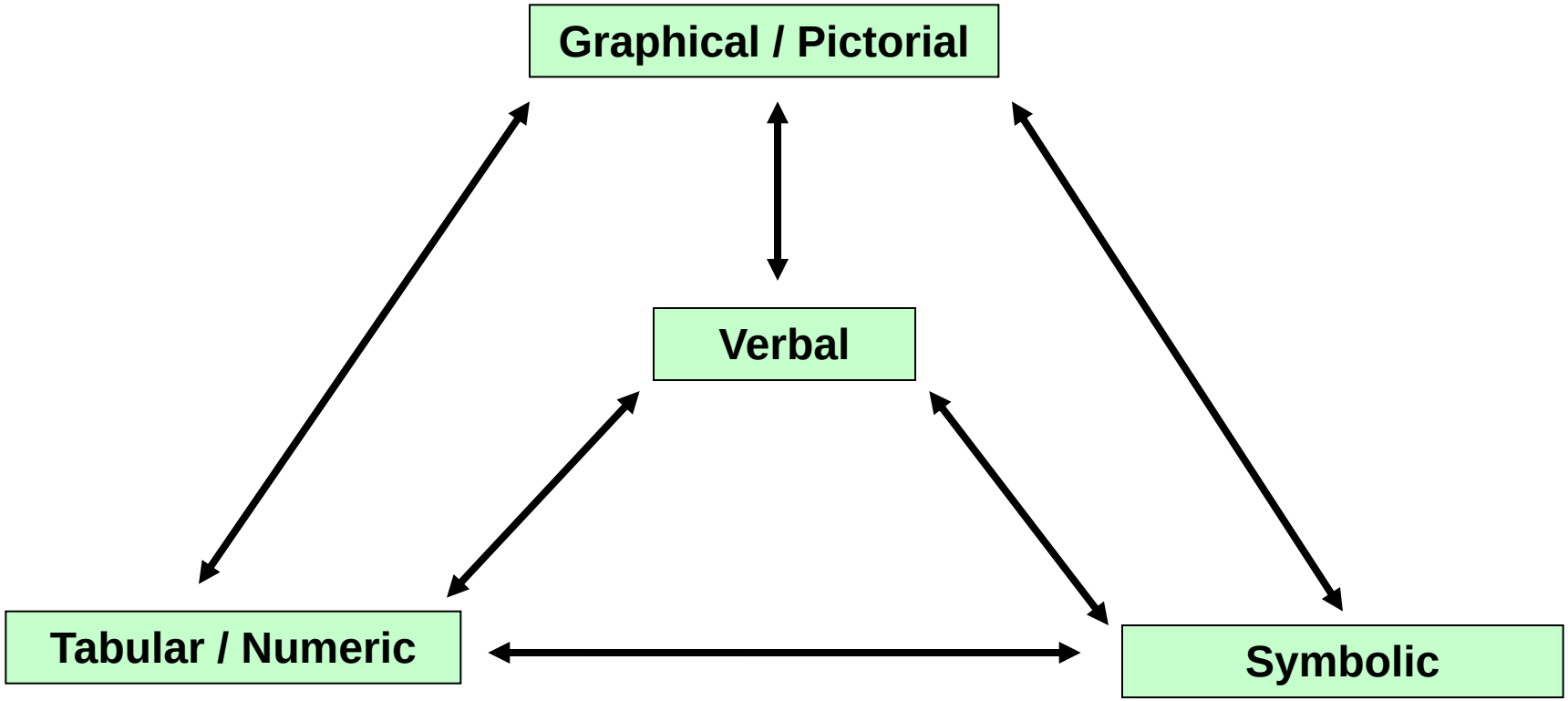
To develop the best model different representations must be investigated.

In this regard technology is very useful.

Focus strongly on construction of a Mathematical Model version for the original problem.



How are the different models interrelated?



Learners need to be able to shift from one model or representation to the other so they can discover new aspects of the concept or situation.

Why different representations?

- Different representations emphasize various aspects or different facets of Mathematics.
- Some learners develop stronger understandings when they see algebraic or symbolic representations.
- Some prefer drawings or graphs while others choose to work with table and numbers.
- It is also true that some learners need to see multiple representations before their sense of concepts takes shape.
- Using different models or representations has the potential to deepen understanding.

How can suitable models be constructed?

Step 1: Identify the problem.
Step 2: Make assumptions
- (a) Identify and classify the variables
- (b) Determine interrelationships between variables

Step 3: Solve the model
Step 4: Verify the model
- (a) Does it address the problem?
- (b) Does it make sense?
- (c) Test it with real-world data

Step 5: Implement the model
Step 6: Maintain the model

NOTE: These are not discrete steps leading nicely to a perfect model and usable results. Simplifying or refining of a model may still be required.

Handshake Problem: Verbal Model

VERBAL MODEL

Twenty people are at a party.

Each person shakes hands with every other person.

How many handshakes are required?

Handshake Problem: Tabular / Numerical Model

VERBAL MODEL

Twenty people are at a party.

Each person shakes hands with every other person.

How many handshakes are required?

TABULAR / NUMERICAL MODEL

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES
1	$0 = 1\times(1-1)\div 2$
2	$1 = 2\times(2-1)\div 2$
3	$1+2 = 3 = 3\times(3-1)\div 2$
4	$1+2+3 = 6 = 4\times(4-1)\div 2$
5	$1+2+3+4 = 10 = 5\times(5-1)\div 2$
6	$1+2+3+4+5 = 15 = 6\times(6-1)\div 2$
7	$1+2+3+4+5+6 = 21 = 7\times(7-1)\div 2$
8	$1+2+3+4+5+6+7 = 28 = 8\times(8-1)\div 2$
9	$1+2+3+4+5+6+7+8 = 36 = 9\times(9-1)\div 2$
10	$1+2+3+4+5+6+7+8+9 = 45 = 10\times(10-1)\div 2$
⋮	⋮
20	$1+2+3+4+5+\cdots+15+16+17+18+19 = 20\times(20-1)\div 2$

Should know :

(Maths Knowledge)

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

and

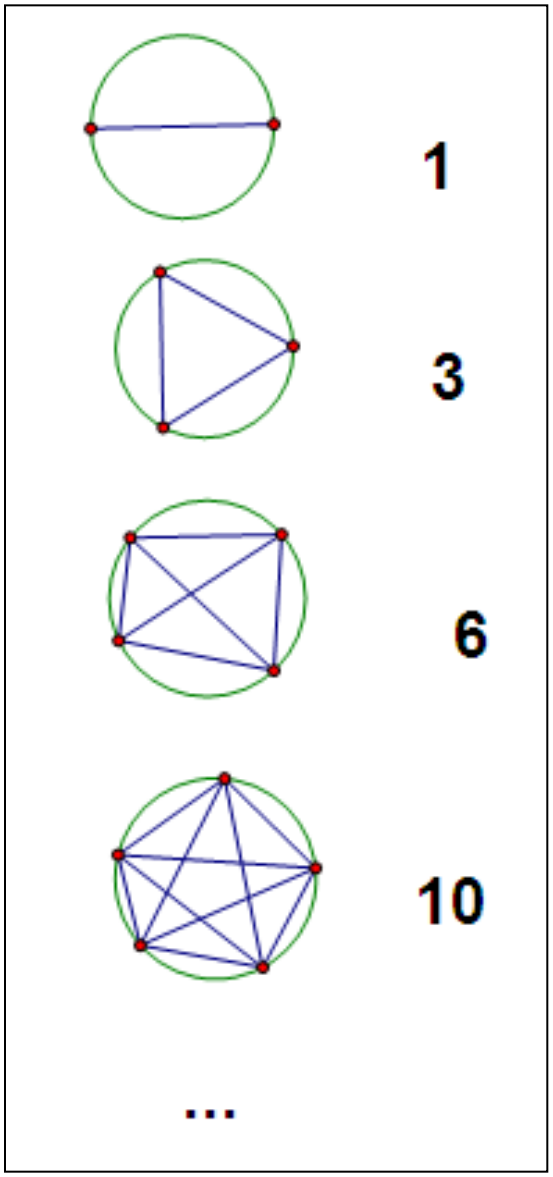
$$0+1+2+3+\cdots \text{ to } n \text{ terms} = \frac{n(n-1)}{2}$$

Handshake Problem: Picture Model

TABULAR / NUMERICAL MODEL

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES
1	0
2	1
3	$1+2=3$
4	$1+2+3=6$
5	$1+2+3+4=10$
6	$1+2+3+4+5=15$
7	$1+2+3+4+5+6=21$
8	$1+2+3+4+5+6+7=28$
9	$1+2+3+4+5+6+7+8=36$
10	$1+2+3+4+5+6+7+8+9=45$
⋮	⋮
20	$1+2+3+4+5+\cdots+15+16+17+18+19=190$

Points on circle: # of people

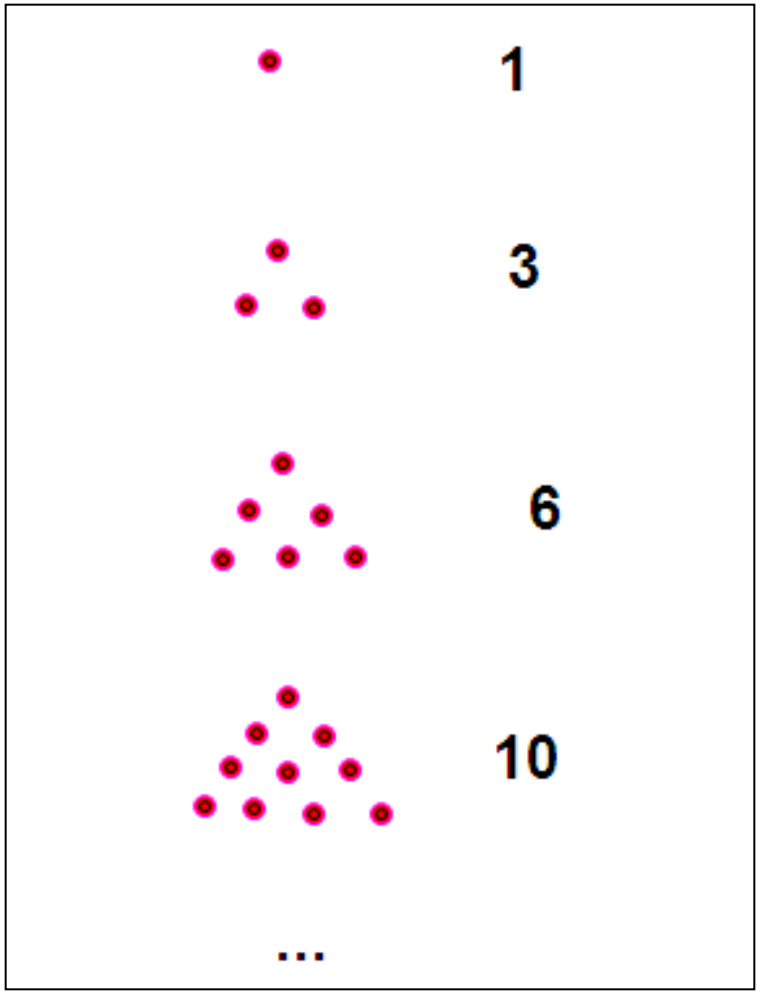


Handshake Problem: Picture Model

TABULAR / NUMERICAL MODEL

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES
1	0
2	1
3	$1+2=3$
4	$1+2+3=6$
5	$1+2+3+4=10$
6	$1+2+3+4+5=15$
7	$1+2+3+4+5+6=21$
8	$1+2+3+4+5+6+7=28$
9	$1+2+3+4+5+6+7+8=36$
10	$1+2+3+4+5+6+7+8+9=45$
⋮	⋮
20	$1+2+3+4+5+\cdots+15+16+17+18+19=190$

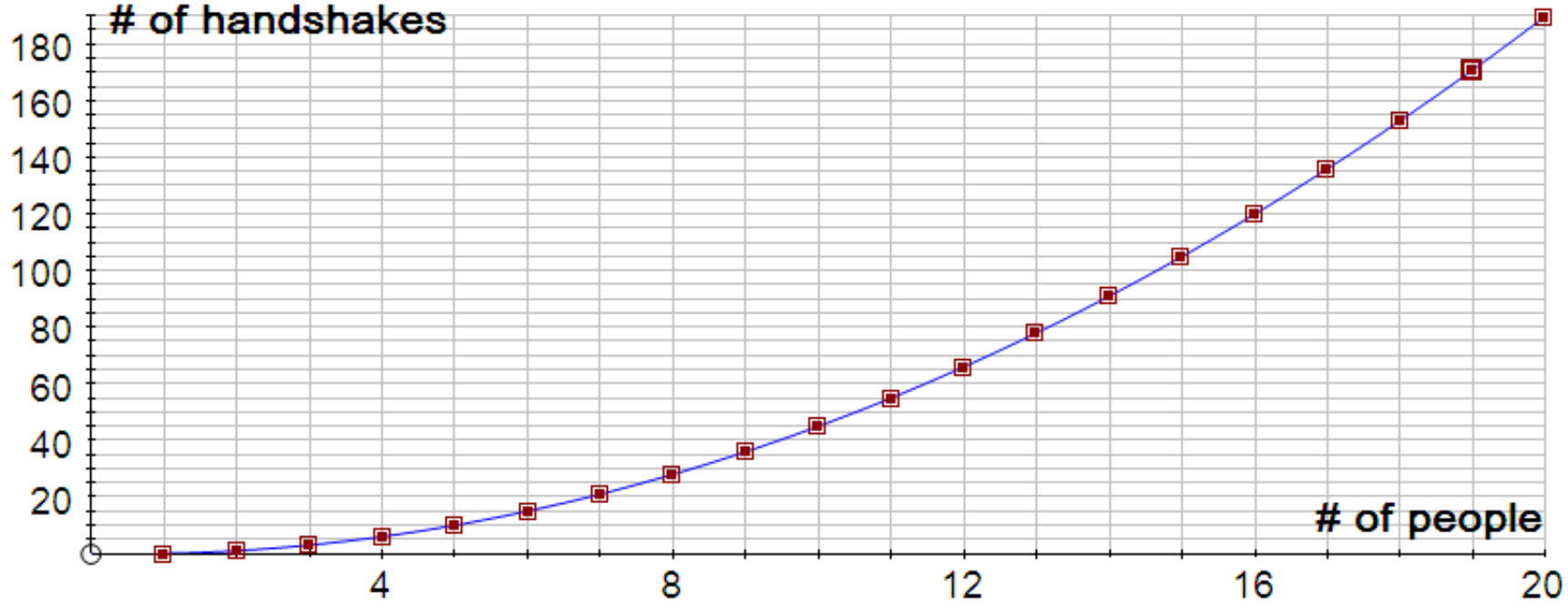
Triangular Numbers



Handshake Problem: Graph Model

TABULAR / NUMERICAL MODEL

# P	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
# H	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190



Handshake Problem: Symbolic Model

TABULAR / NUMERICAL MODEL

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES
1	$0 = 1\times(1-1)\div 2$
2	$1 = 2\times(2-1)\div 2$
3	$1+2 = 3 = 3\times(3-1)\div 2$
4	$1+2+3 = 6 = 4\times(4-1)\div 2$
5	$1+2+3+4 = 10 = 5\times(5-1)\div 2$
6	$1+2+3+4+5 = 15 = 6\times(6-1)\div 2$
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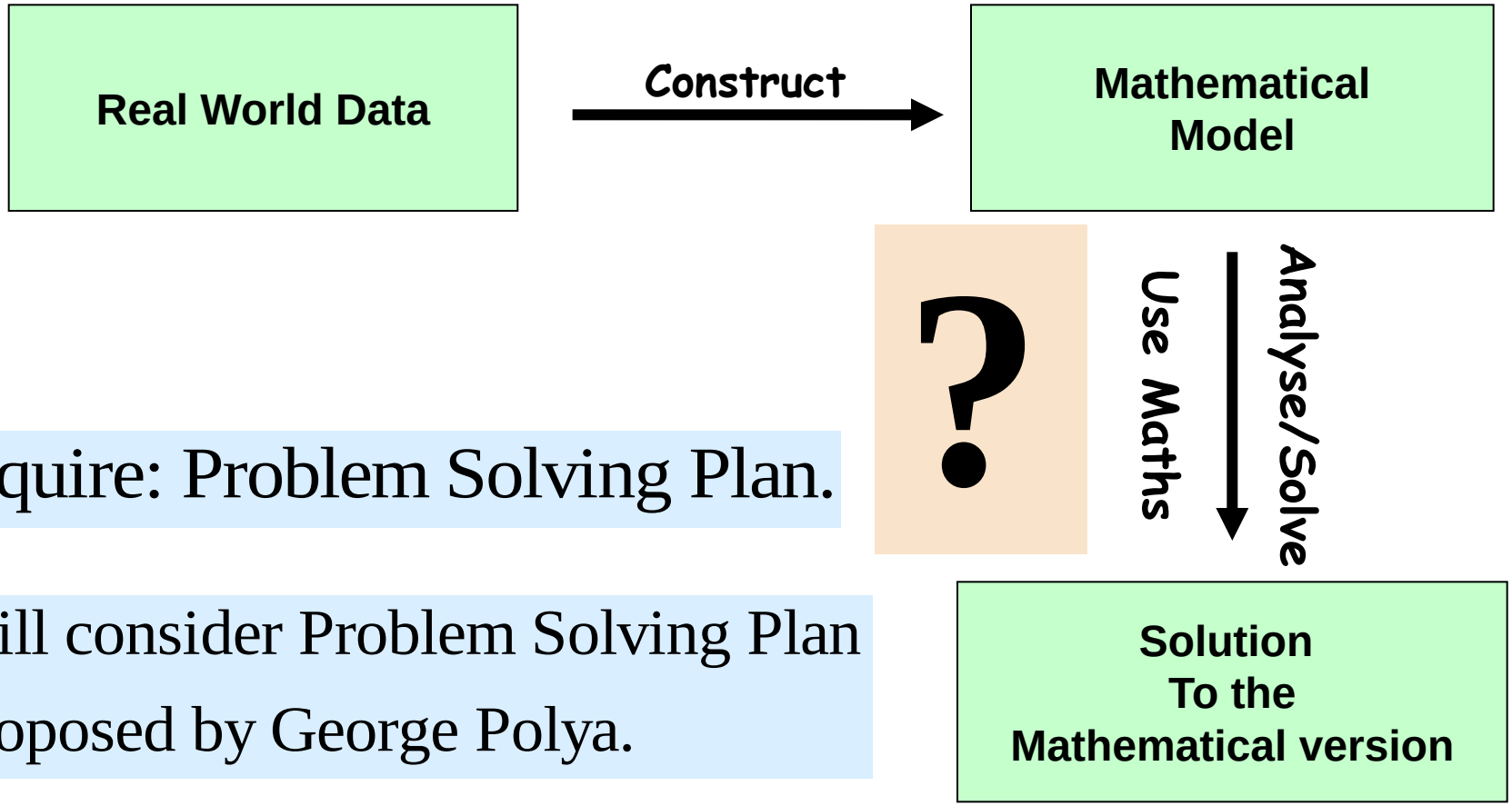
10	$1+2+3+4+5+6+7+8+9 = 45 = 10\times(10-1)\div 2$
⋮	⋮
20	$1+2+3+4+5+\cdots+15+16+17+18+19 = 20\times(20-1)\div 2$

$$H(P) = \frac{P(P-1)}{2}$$

MATHEMATICIAN

$${}^{P}C_{2}$$

Solving the constructed Mathematical Model.



- Require: Problem Solving Plan.

- Will consider Problem Solving Plan proposed by George Polya.

George Polya: The Father of Modern PS

- **Born in Hungary in 1887.**
- **Wrote over 250 mathematical papers and three books that promote PS.**
- **His most famous book, *How to solve it*, which has been translated into 15 languages.**
- **Introduced a four-step approach to PS**
  - **Understand the problem**
  - **Devise a plan**
  - **Carry out the plan**
  - **Look back**
- **Together with strategies, which are helpful in solving problems.**
- **He died in 1985, leaving mathematics with the important legacy of teaching PS.**

Step 1: Understand the Problem

- **Do you understand the words?**
- **Can you restate the problem in your own words?**
- **Do you know what is given?**
- **Do you know what the goal is?**
- **Is there enough information?**
- **Is there extraneous information?**
- **Is this problem similar to another problem you have solved?**

Step 2: Devise a Plan
Strategy (Artful means to an end)

1. Guess and test.
2. Use a variable.
3. Draw a picture.
4. Look for a pattern.
5. Make a list.
6. Solve a simpler problem.
7. Draw a diagram.
8. Use direct reasoning.
9. Use indirect reasoning.
10. Use properties of numbers.
11. Solve an equivalent problem.
12. Work backward.
13. Use cases.
14. Solve an equation.
15. Look for a formula.
16. Do a simulation.
17. Use a model.
18. Use dimensional analysis.
19. Identify sub-goals.
20. Use co-ordinates.
21. Use symmetry.
22. Many more.

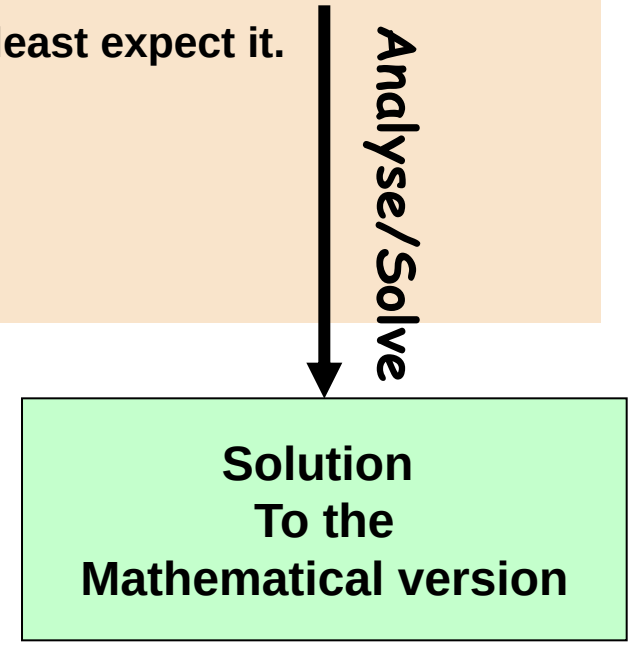
- Often use a combination of strategies.
- Will only consider a few – rest for you to explore.

Step 3: Carry out the Plan

- **Implement the strategy or strategies that you have chosen.**
  - **until the problem is solved**
  - **or until a new course of action is suggested.**
- **Give yourself a reasonable amount of time in which to solve the problem.**
  - **If you are not successful,**
    - **seek hints from others**
    - **or put the problem aside for a while.**
    - **You may have a flash of insight when you least expect it.**
- **Do not be afraid of starting over.**
  - **Often, a fresh start and a new strategy**
  - **will lead to success.**

- Focus strongly on
  - Analysing and
  - Solving the

Mathematical Model Version of the problem.

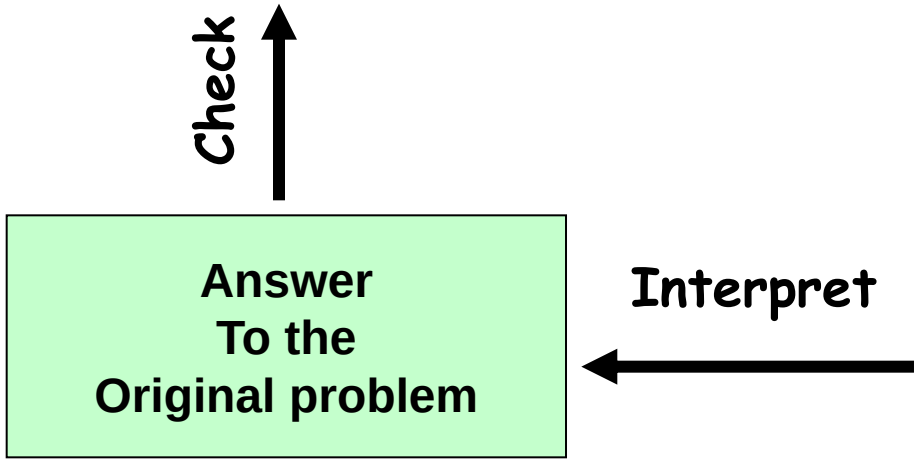


Step 4: Look Back

- **Is your solution correct?**
- **Does your answer satisfy the statement of the problem?**
- **Can you see an easier solution?**
- **Can you see how you can extend you solution to more general cases?**
- **Can you generate your own but similar examples?**

- Focus strongly on
  - Interpreting and
  - Checking

Mathematical Solution



Tutorial 1: Modelling and Problem Solving

Salesperson has two choices as how she will be paid.

<u>Plan A</u>: R50 for every computer system sold.

<u>Plan B</u>: R250 per week + R25 for every system sold.

- **What would you recommend?**
- **Which strategy will you use?**
- **How will you model the problem?**

PAUSE DVD

- Do Tutorial 1
- Then View Solutions

Tutorial 1: Suggested Solution
Strategy 1: Guess and check (Construct a table)

Sales	Plan A	Plan B	Reflection
0	0	250	Plan B the best $\Rightarrow$ Do nothing and earn R250 (For how long?)
1	50	250+25=275	Plan B best $\rightarrow$ Earns R225 more
2	100	$250+2\times 25=300$	Difference now R200 $\rightarrow$ Plan A cathing up.
⋮	⋮	⋮	⋮
9	450	$250+9\times 25=475$	Plan B just the best
10	500	$250+10\times 25=500$	No difference between the two plans.
11	550	$250+11\times 25=525$	Plan A now the better plan.
12	600	550	Plan A the best.
⋮	⋮	⋮	⋮

Tutorial 1: Suggested Solution
Strategy 2: Use algebra

Plan A: R50 for every computer system sold.

Plan B: R250 per week + R25 for every system sold.

y: Weekly salary

x: Number of systems sold

Plan A: $y = 50x$

Plan B: $y = 250 + 25x$

Solve simultaneously: $50x = 250 + 25x \Rightarrow 25x = 250 \Rightarrow x = 10$

Reflection!

Tutorial 1: Suggested Solution
Strategy 3: Draw a graph

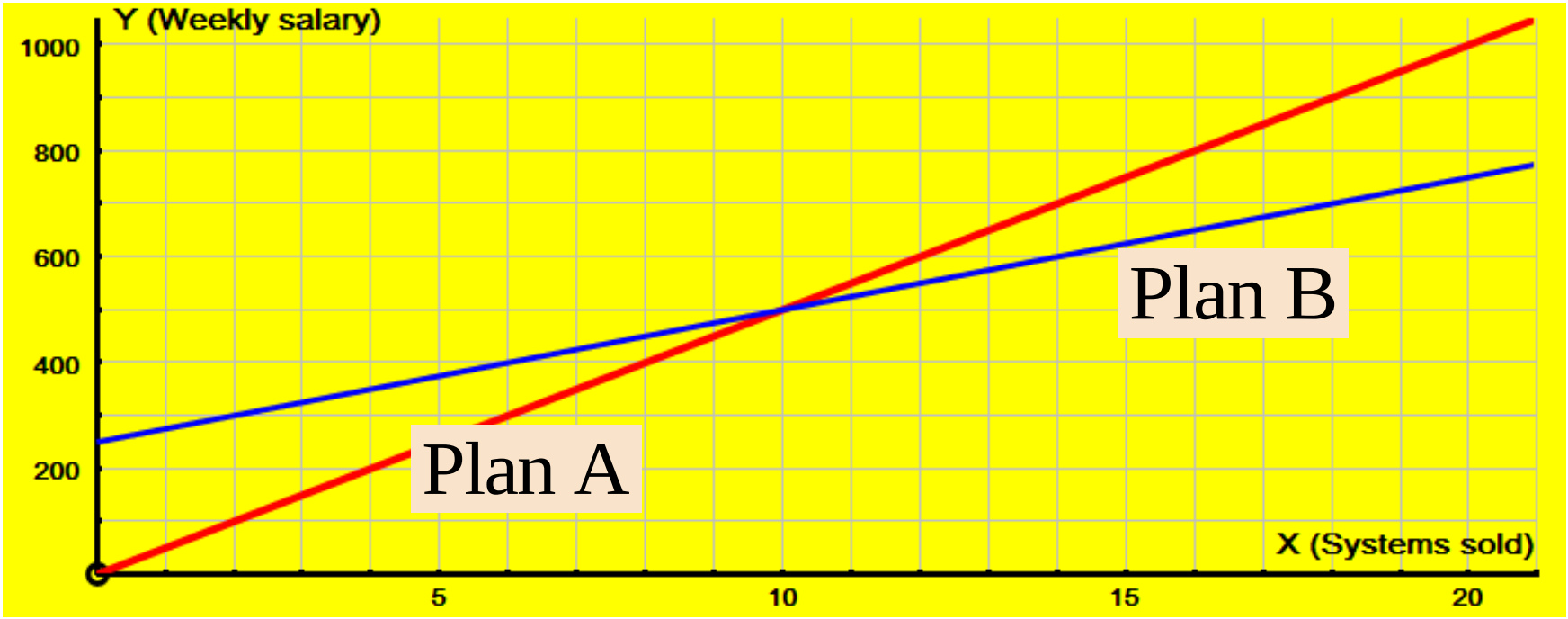
y : Weekly salary

x : Number of systems sold

Plan A: $y = 50x$

Plan B: $y = 250 + 25x$

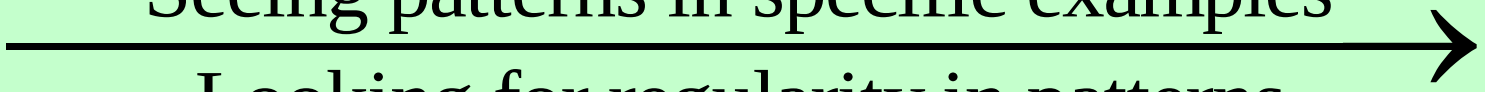
Reflection!



Inductive Reasoning

Consider specific examples

Seeing patterns in specific examples

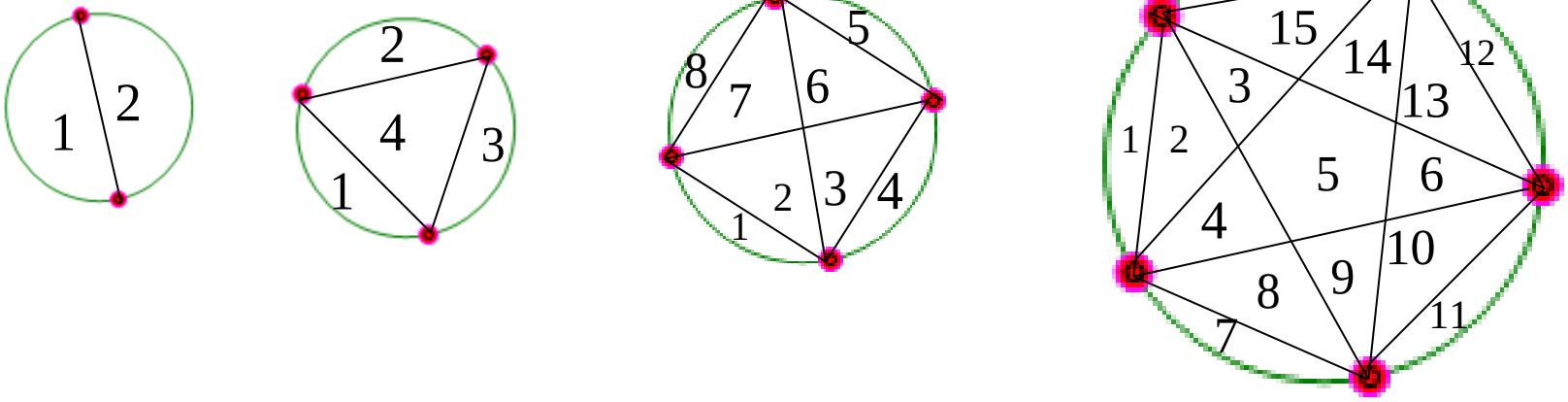


Looking for regularity in patterns

Coming to a general conclusion

Inductive reasoning
Be careful!

Consider the following problem:
Join 2, 3, 4, … points on a circle to obtain the maximum number of regions.



# of points (P)	2	3	4	5
# of regions (R)	2	4	8	16

$$R(P) = 2^{P-1}$$

Complete and extend the table

# of points (P)	2	3	4	5	6
# of regions (R)	2	4	8	16	**32**

$$R(P) = 2^{P-1}$$

$$\Rightarrow R(6) = 2^5 = 32$$

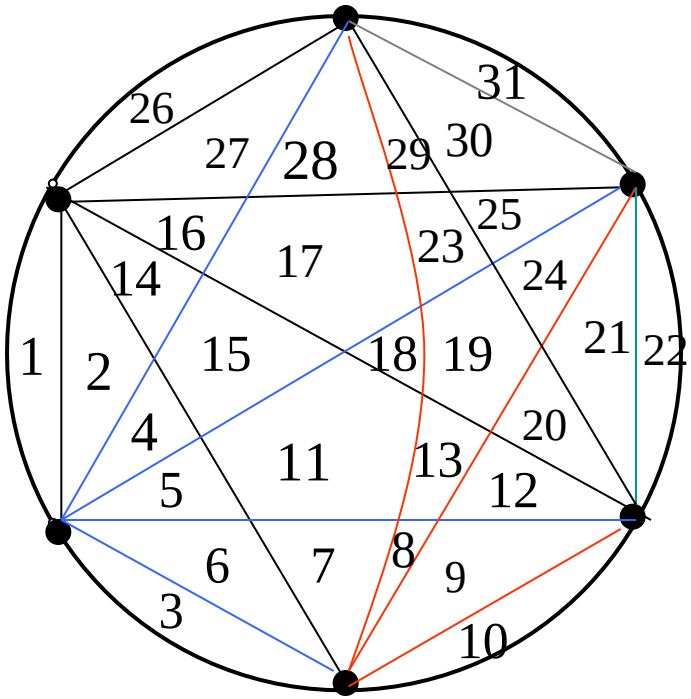
Important caution connected to inductive reasoning:

- **Will there be 32 regions if we utilize 6 points?**
- **Danger of simplified assumptions.**
- **Investigate sufficient specific examples.**
- **Before you generalize.**

Be careful – Some patterns do not always hold.

Further investigation required

Consider a circle on which 6 points are joined by all possible lines in such a way that the maximum number of regions is obtained.



# of points	# of regions
1	0
2	2
3	4
4	8
5	16
6	**31**

$$R(P) \neq 2^{P-1}$$

- More investigations needed! (Cyclic Nature)

How many regions if 7 points are placed on the circle?

Assume that the pattern will now hold.

Dangerous over simplified assumption!

Use method of finite differences to find $R(7)$:

P	2	3	4	5	6	7
$R(P)$	2	4	8	16	31	**57**
1st Difference	2	4	8	15	**26**	
2nd Difference	2	4	7	**11**		
3rd Difference	2	3	**4**			
4th Difference	1	**1**				

# of points P	# of regions $R(P)$
1	0
2	2
3	4
4	8
5	16
6	**31**
7	**57?**

Investigate by means of an additional sketch!

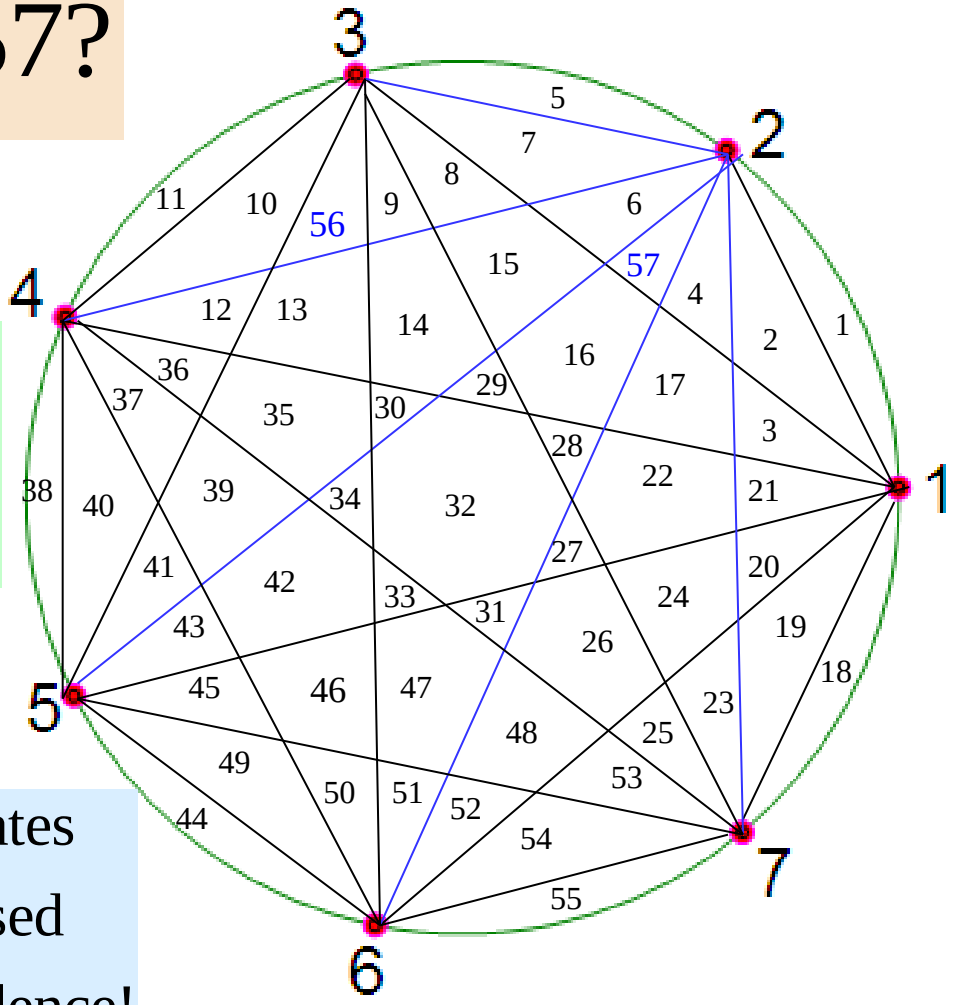
ADDITIONAL

Simplified assumption : 4^{th} Differences will be constant.

Additional Inductive Investigation

Is $R(7) = 57$?

Can relationship between P and $R(P)$ be generalized?



$\therefore$ Experiment correlates with prediction based on Theoretical Evidence!

General Formulae

P	2	3	4	5	6	7
$R(P)$	2	4	8	16	31	57

Assumption: 4th Differences Constant

$$\Rightarrow R(P) = aP^4 + bP^3 + cP^2 + dP + e$$

$$16a + 8b + 4c + 2d + e = 2 \quad \cdots(1)$$
$$81a + 27b + 9c + 3d + e = 4 \quad \cdots(2)$$
$$256 + 64b + 16c + 4d + e = 8 \quad \cdots(3)$$
$$625a + 125b + 25c + 5d + e = 16 \quad \cdots(4)$$
$$1296a + 216b + 36c + 6d + e = 31 \quad \cdots(5)$$

$$(2)-(1): 65a + 19b + 5c + d = 2 \quad \cdots(6)$$
$$(3)-(2): 175a + 37b + 7c + d = 4 \quad \cdots(7)$$
$$(4)-(3): 369a + 61b + 9c + d = 8 \quad \cdots(8)$$
$$(5)-(4): 671 + 91b + 11c + d = 15 \quad \cdots(9)$$

$$(7)-(6): 110a + 18b + 2c = 2 \quad \cdots(10)$$
$$(8)-(7): 194a + 24b + 2c = 4 \quad \cdots(11)$$
$$(9)-(8): 302a + 30b + 2c = 7 \quad \cdots(12)$$

$$(11)-(10): 84a + 6b = 2 \quad \cdots(13)$$
$$(12)-(11): 108 + 6b = 3 \quad \cdots(14)$$

$$(14)-(13): 24a = 1 \Rightarrow a = \frac{1}{24}$$

Further more:

$$b = -\frac{1}{4};\ c = \frac{23}{24};\ d = -\frac{3}{4};\ e = 1$$

$$\therefore R(P) = \frac{P^4}{24} - \frac{P^3}{4} + \frac{23P^2}{24} - \frac{3P}{4} + 1 = \frac{P^4 - 6P^3 + 23P^2 - 18P + 24}{24}$$

Test the validity of General Formulae

$$\therefore R(P)=\frac{P^4}{24}-\frac{P^3}{4}+\frac{23P^2}{24}-\frac{3P}{4}+1=\frac{P^4-6P^3+23P^2-18P+24}{24}$$

$$R(6)=\frac{6^4-6\times6^3+23\times6^2-18\times6+24}{24}=31$$

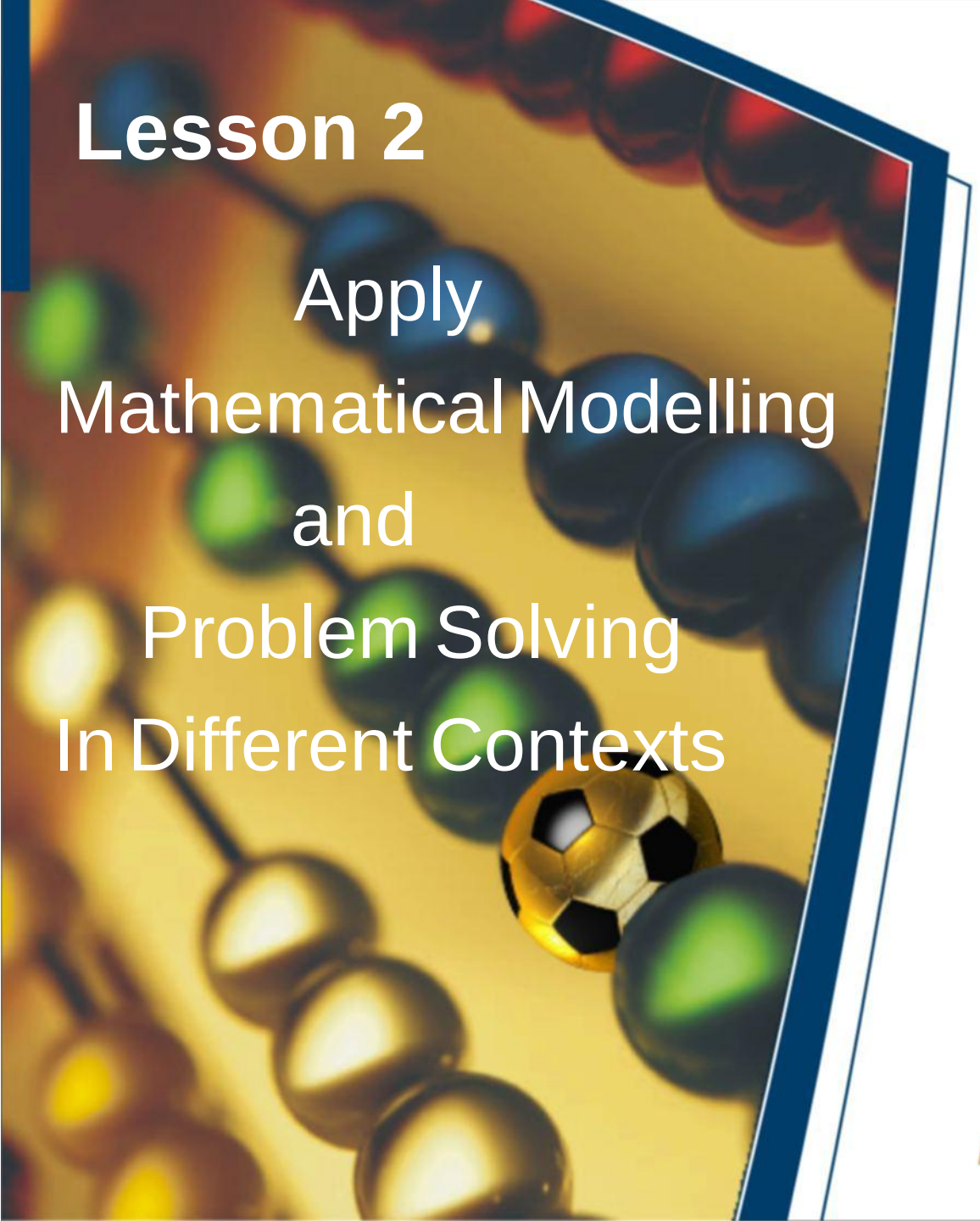
Similarily: $R(7)=57$ and $R(8)=99$

Can we be sure that $R(8)=99$?

Further research needed (Make a new sketch!)

Sketches sooner or later not practical/possible any more.

But the Cyclic Nature must continue until we absolutely sure about the correctness of the generalization!



Lesson 2

Apply Mathematical Modelling and Problem Solving In Different Contexts

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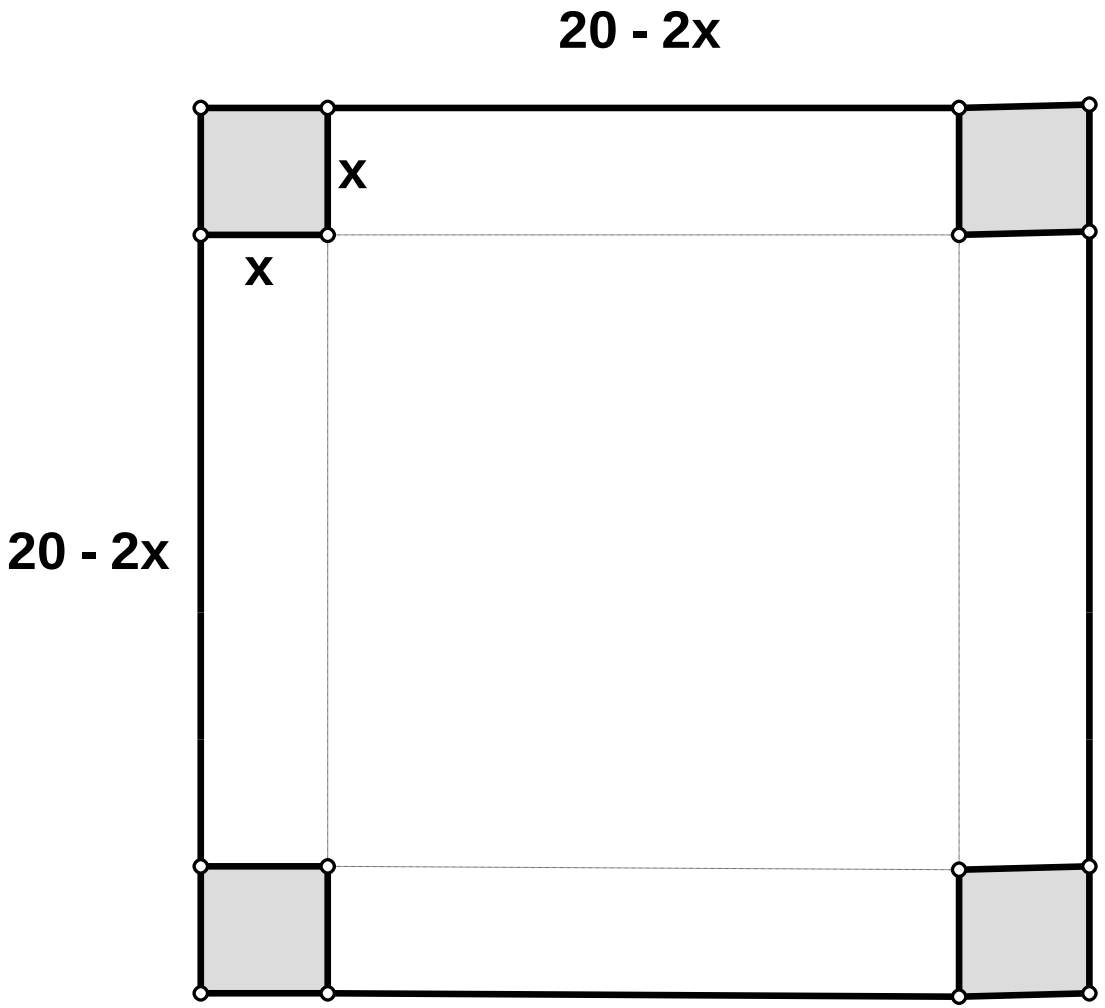
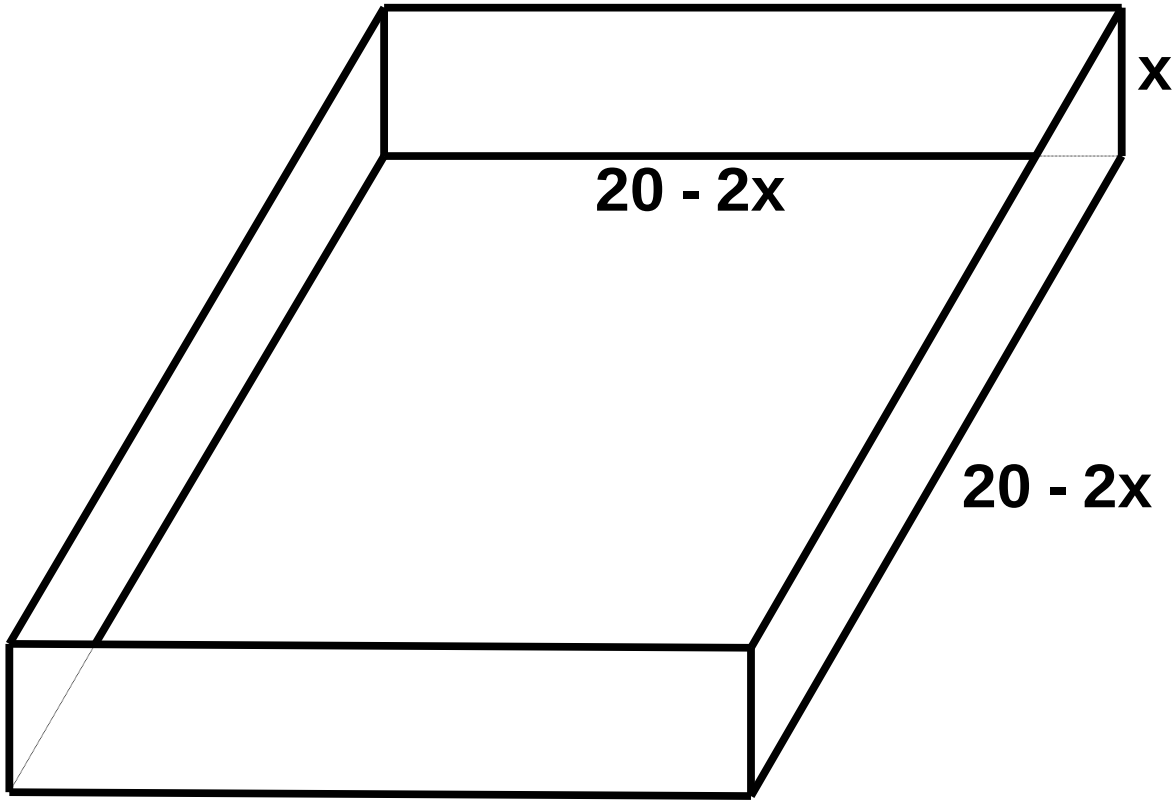
Maximum volume of box?

A box, without a lid, is to be constructed from a piece of square cardboard measuring 20 cm by 20 cm. The box is constructed by cutting out square corners and folding up the resulting sides. What is the maximum volume of a box that can be formed in this way?

Pictorial model of the construction material (Net).



From Pictorial model to Symbolic model.



$$V(x) = x(20 - 2x)^2$$

From Equation Model to Table Model

$$V(x) = x(20 - 2x)^2$$

Corner dimension in cm	Volume in cm^3
1	324
2	**512**
3	**588**
4	**576**
5	500
6	384

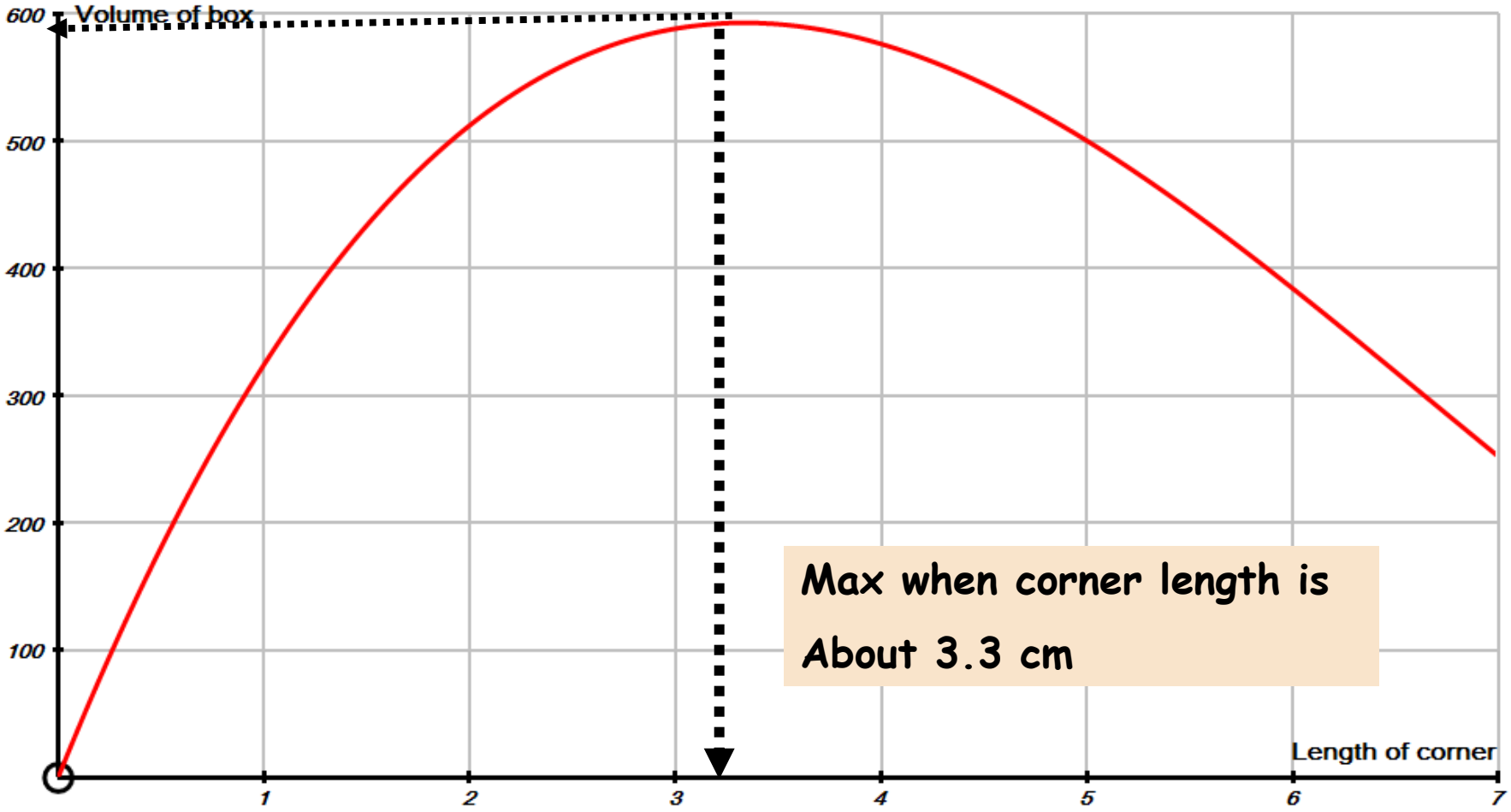
Refine the Table Model.

Corner	Volume	Corner	Volume
2.1	512	3.1	590
2.2	524	3.2	591.872
2.3	535	3.3	592.548
2.4	545	3.4	592.416
2.5	554	3.5	591
2.6	562	3.6	589
2.7	569	3.7	587
2.8	575	3.8	584
2.9	580	3.9	580
3.0	588	4.0	576

Construct a Graph Model.

Max close to 600

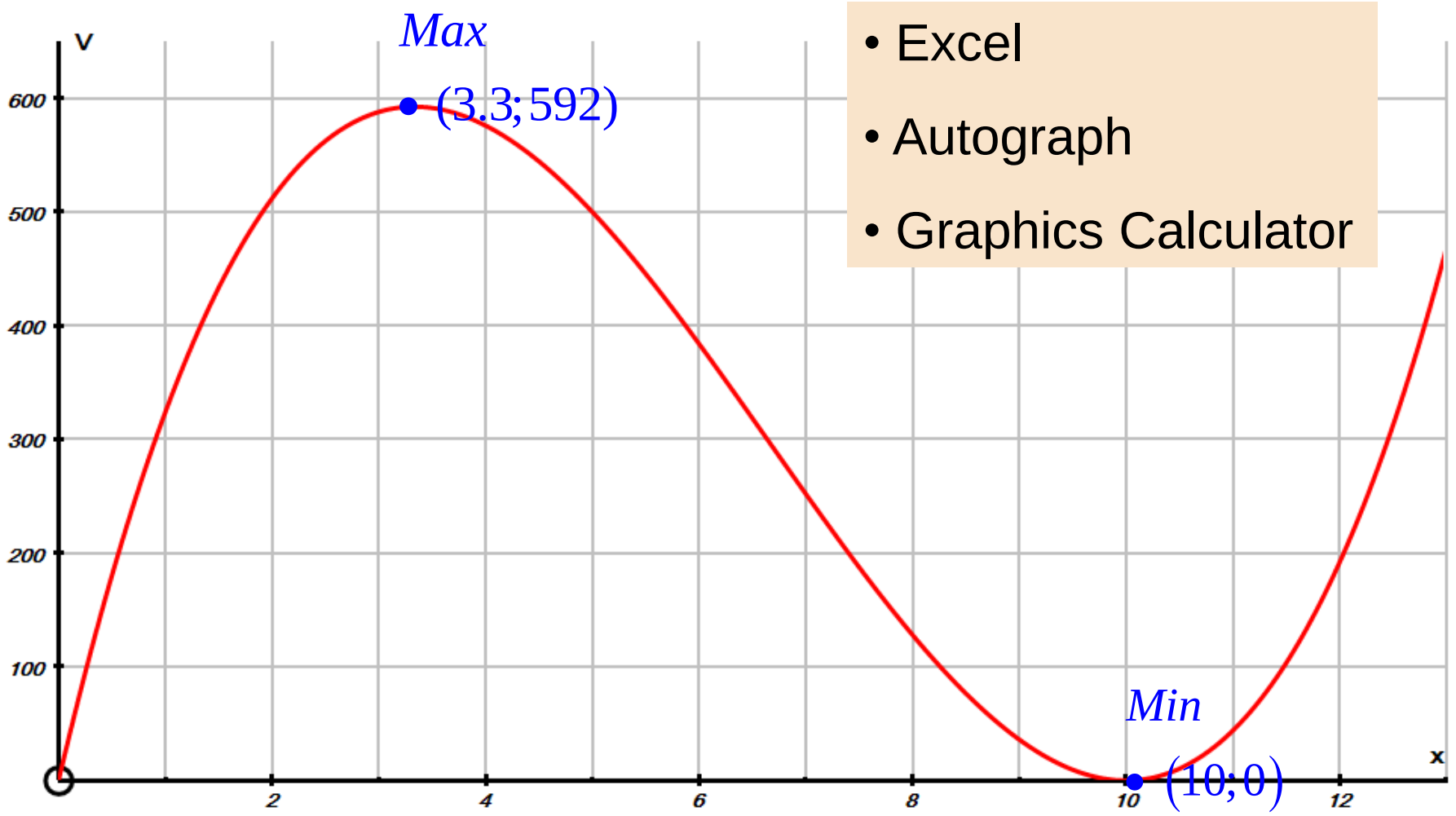
$$V(x) = 400x - 80x^2 + 4x^3$$



Max when corner length is

About 3.3 cm

Spotlight on Technology.

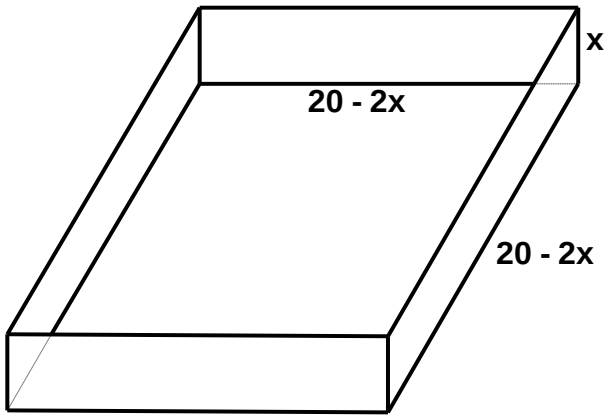


- Excel
- Autograph
- Graphics Calculator

Solution to problem?

- Maximum volume about 592 cm^3
- Corner dimension about 3.3 cm by 3.3 cm
- Box about 3.3 cm x 13.4 cm x 13.4 cm
- Needs refinement (Grade 12 – Differentiation)

Mathematics: Calculus



$$V(x) = x(20-2x)^2$$
$$= 400x - 80x + 4x^3$$

$$V'(x) = 400 - 160x + 12x^2$$

$$V'(x) = 0 \Rightarrow 400 - 160x + 12x^2 = 0$$
$$\Rightarrow 100 - 40x + 3x^2 = 0$$
$$\Rightarrow (3x-10)(x-10) = 0$$

$\therefore$ Critical numbers are $10/3$ or 10

$$V''(x) = -160 + 24x$$

$$V''(10/3) = -160 + 80 < 0$$

Volume max when $x = 10/3$

$$\text{Max volume} = V(10/3) = 10/3\,(20 - 20/3)^2$$
$$= 10/3 \times (40/3)^2 = 16000/27 = 592\,16/27$$

$$V''(10) = -160 + 240 > 0$$

Volume min when $x = 10$

Tutorial 2: Maximum Volume of Box

A box, without a lid, is to be constructed from a piece of rectangular cardboard measuring 20 cm by 30 cm. The box is constructed by cutting out square corners and folding up the resulting sides.

1. Sketch a net to construct the box.
2. Determine the volume, $V(x)$, in terms of x, side length of corner square.
3. Determine, by means of a table model, when the volume will be a maximum.

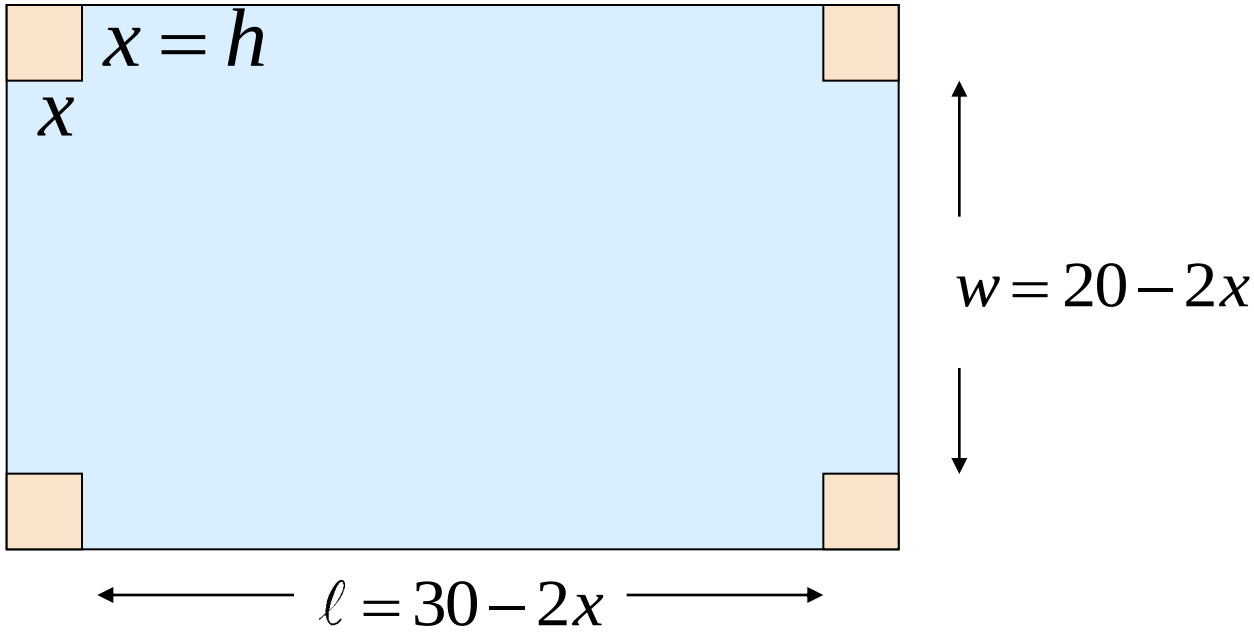
PAUSE DVD

- Do Tutorial 2
- Then View Solutions

Tutorial 2 Problem 1: Suggested Solution

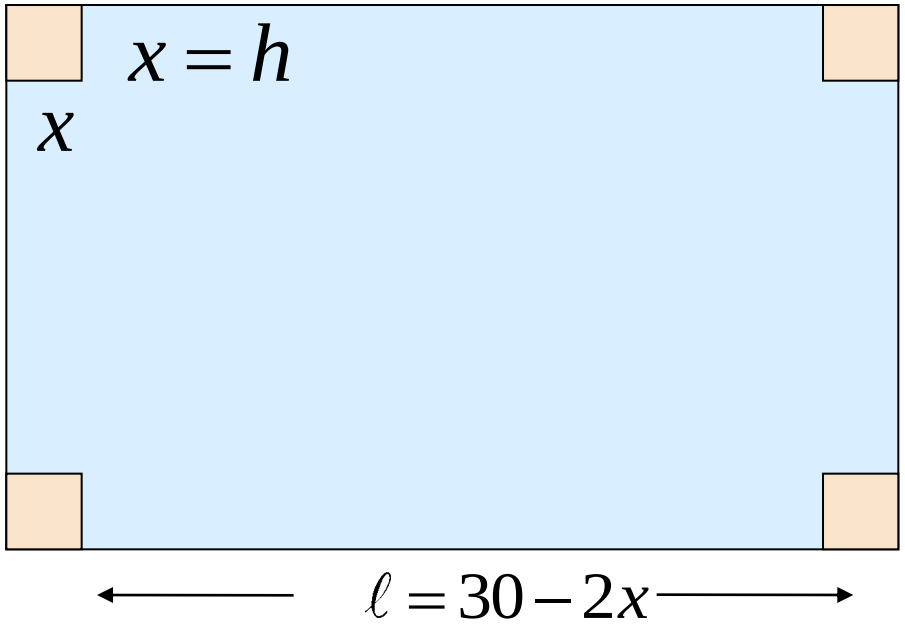
A box, without a lid, is to be constructed from a piece of rectangular cardboard measuring 20 cm by 30 cm. The box is constructed by cutting out square corners and folding up the resulting sides.

1. Sketch a net to construct the box.



Tutorial 2 Problem 2: Suggested Solution

2. Determine the volume, $V(x)$, in terms of x, side length of corner square.



$$\therefore V(x) = \ell \times w \times h$$

$$\therefore V(x) = x(30 - 2x)(20 - 2x)$$

Tutorial 2 Problem 3: Suggested Solution

3. Determine, by means of a table model, when the volume will be a maximum.

$$\therefore V(x) = x(30-2x)(20-2x)$$

x	0	1	2	3	4	5	6	7	8	9	10
$V(x)$	0	504	832	1008	1056	1000	864	672	448	216	0

x	3	3,2	3,4	3,6	3,8	4	4,2	4,4	4,6	4,8	5
$V(x)$	1008	1027	1041	1051	1055	1056	1052	1045	1033	1018	1000

x	3,80	3,85	3,90	3,95	4,00	4,05	4,10	4,15	4,20	4,25	4,30
$V(x)$	1055,49	1056,02	1056,28	1056,27	1056,00	1055,47	1054,68	1053,64	1052,35	1050,81	1049,03

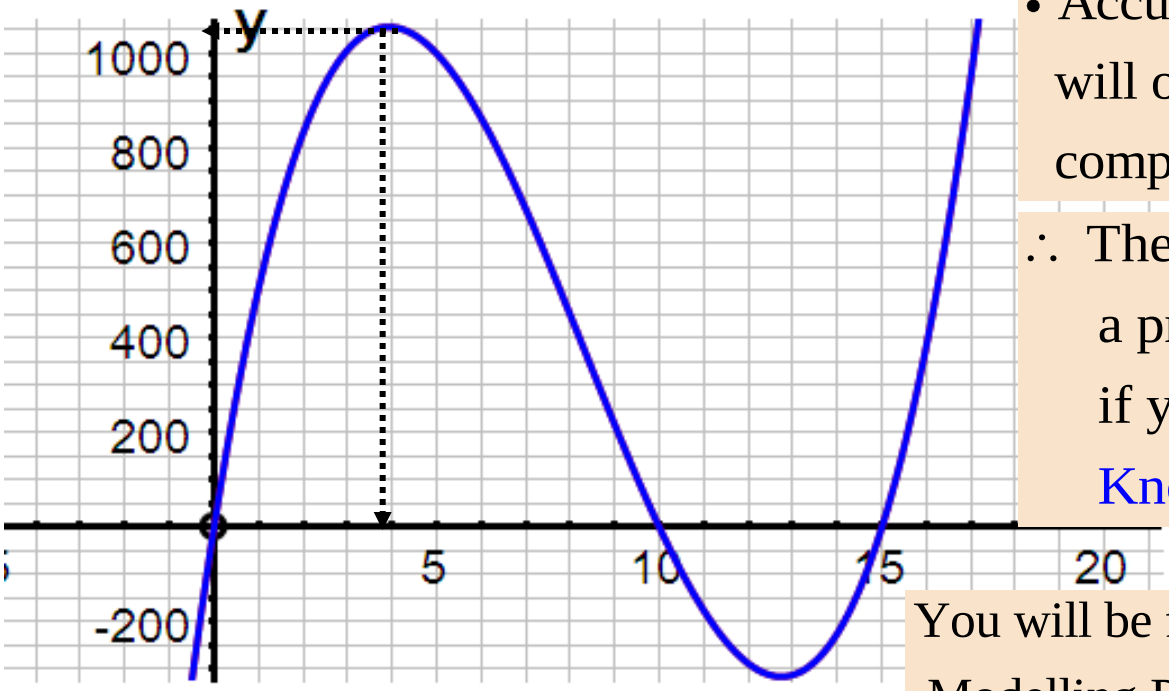
Max volume when $x \in [3,85; 3; 95]$

More refinement!

Tutorial 2 Problem 3: Graphical Suggested Solution

$$\therefore V(x) = x(30-2x)(20-2x)$$
$$= 4x^3 - 100x^2 - 600x$$

Max volume when $x \in [3,85;3;95] \wedge V \simeq 1056$



- You will understand and be able to model this problem by means of a graph once we have completed the DVD on Cubic Functions.
- Accurate solution to this problem will only be possible once you completed the Calculus DVD.

∴ The way you model and solve a problem can only be refined if your available Mathematical Knowledge is uplifted!

You will be faced with more Mathematical Modelling Problem Solving situations in the section dealing with Linear Programming.

End of the DVD on Mathematics Modelling and Problem Solving

REMEMBER!

- Consult text-books for additional examples.
- Attempt as many as possible other similar examples on your own.
- Compare your methods with those that were discussed in the DVD.
- Repeat this procedure until you are confident.
- Do not forget:

Practice makes perfect!