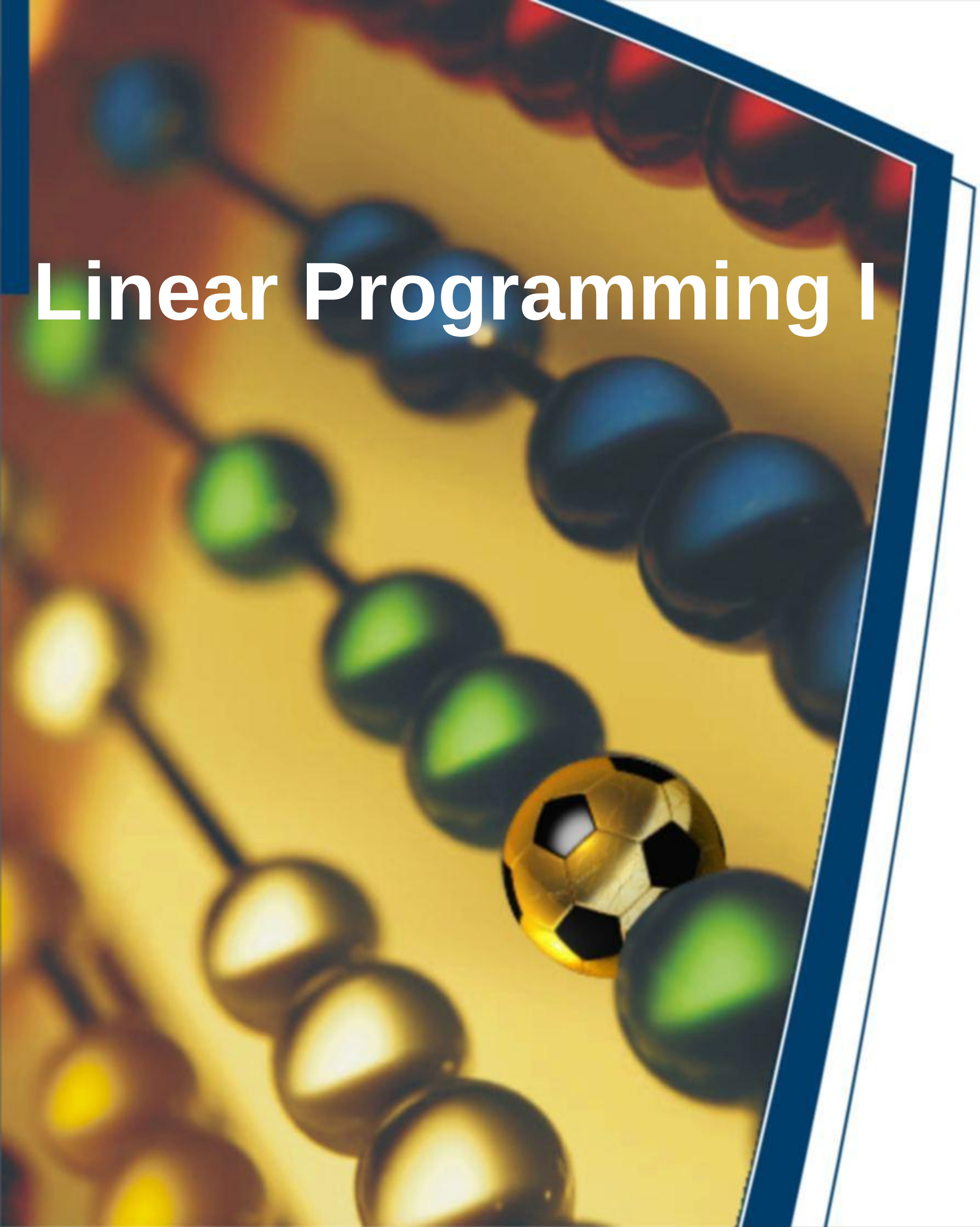

Linear Programming I

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Outcomes for this DVD

In this DVD we will:

- Interpret and graph linear inequalities in the Cartesian Plane

  Lesson 1

- Focus on Modelling of Linear Programming Problems

  Lesson 2

Lesson 1

Interpret and Graph Linear Inequalities

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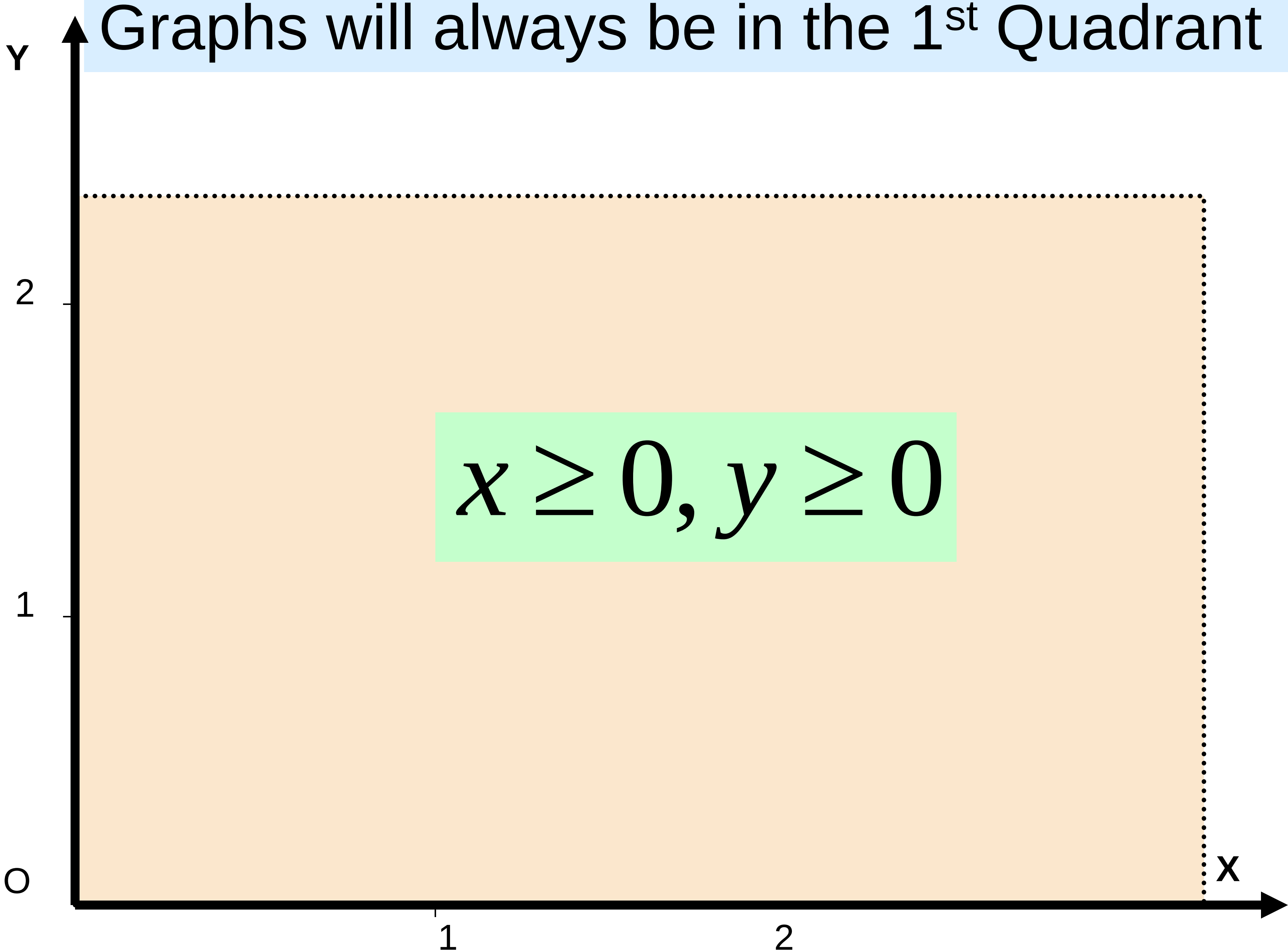
Linear Programming: A General Framework

- Linear Programming is used to **find the best solution** to a problem
- The problem can always be described as an **objective function** that has to be **maximized or minimized** subject to a **set of constraints**
- The **constraints** associated with a linear programming problem **are always linear inequalities**
- The **objective function** will always **depend on two independent variables** that will assume **positive values** only
- The **solution** of a linear programming problem can always be **found graphically** in the **first quadrant** of the Cartesian Plane

Graphing Linear Inequalities: Example 1

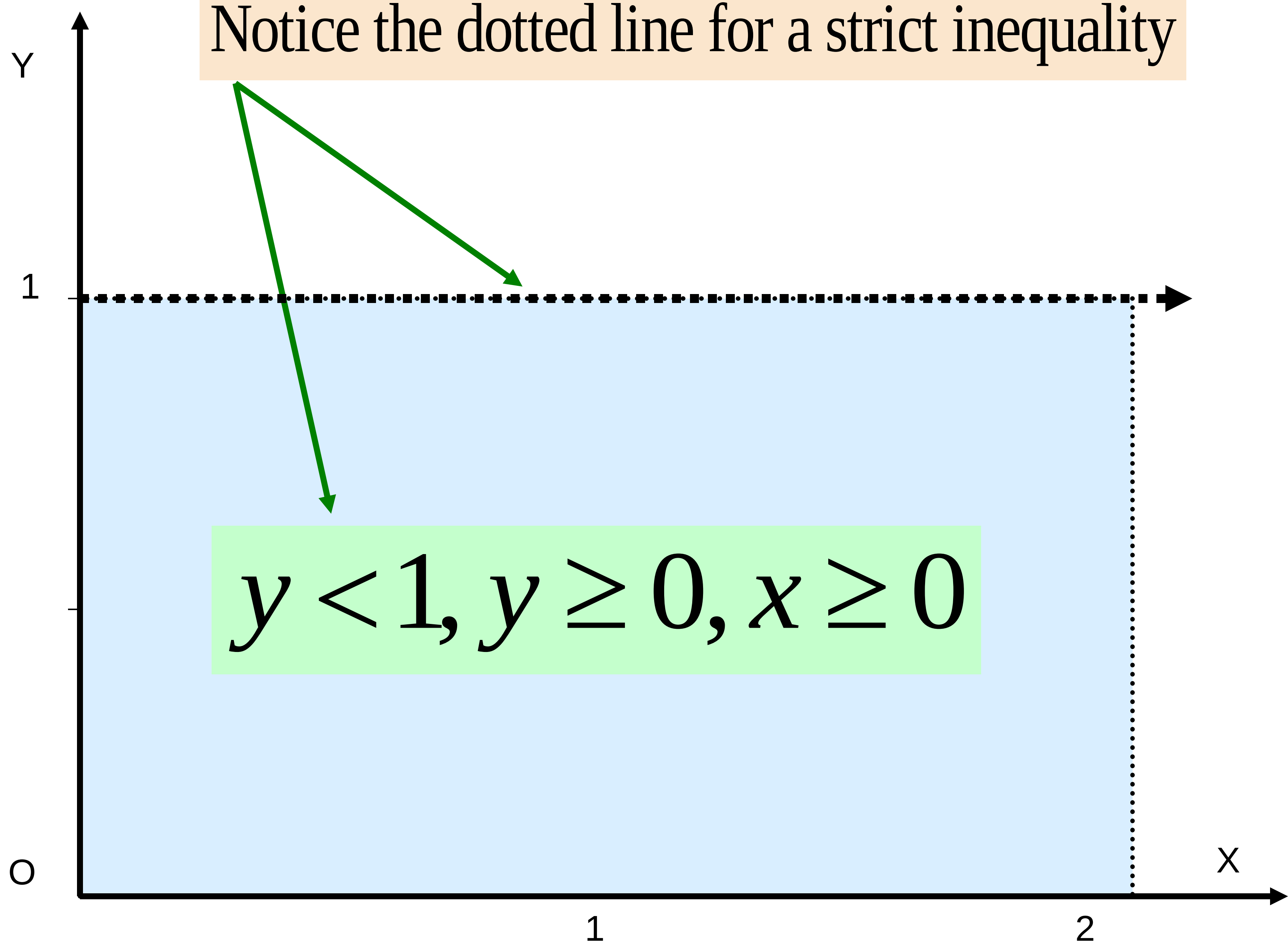
Graphs will always be in the 1st Quadrant



$$x \geq 0, y \geq 0$$

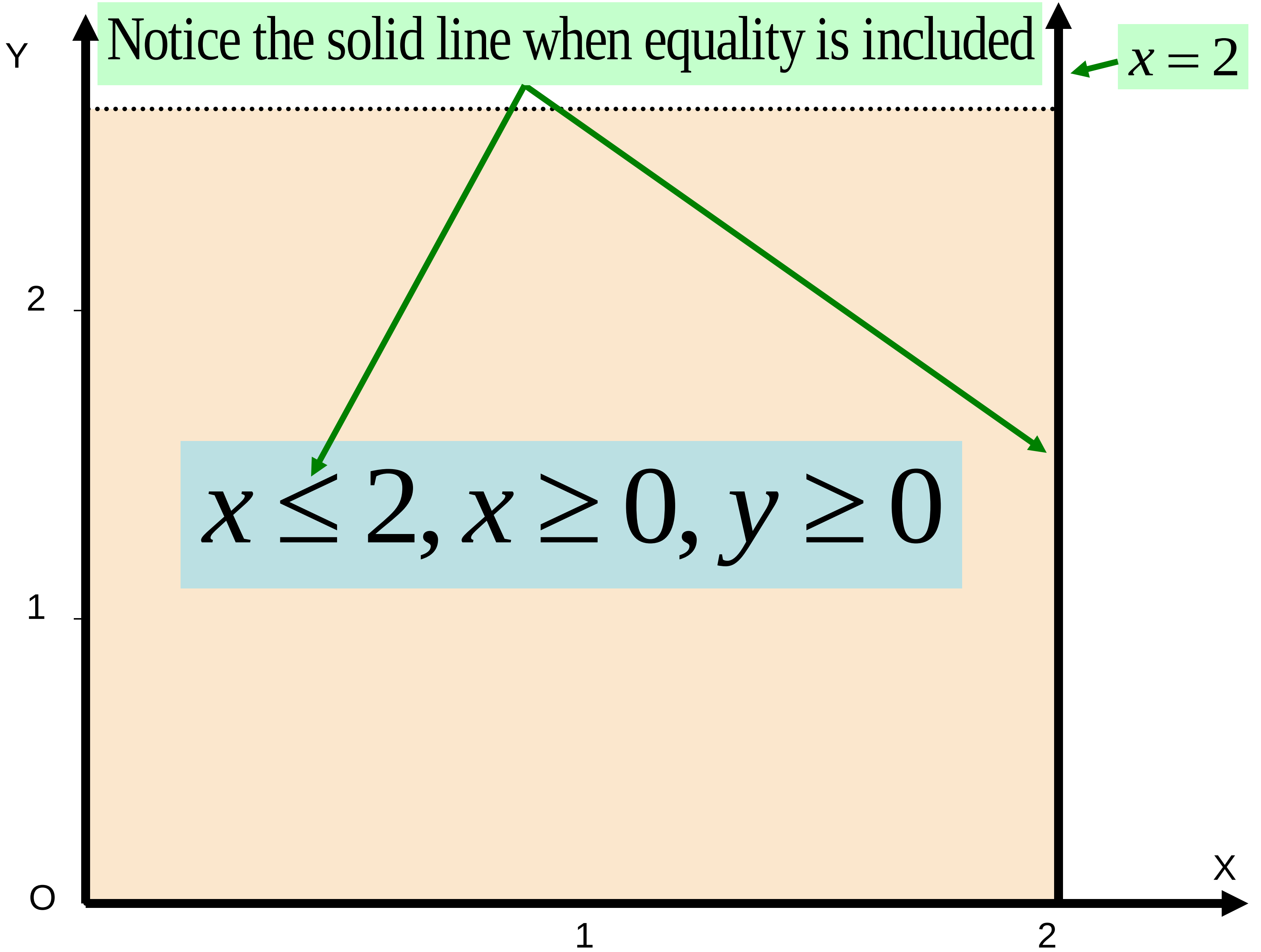
Graphing Linear Inequalities: Example 2

Notice the dotted line for a strict inequality



$$y < 1, y \geq 0, x \geq 0$$

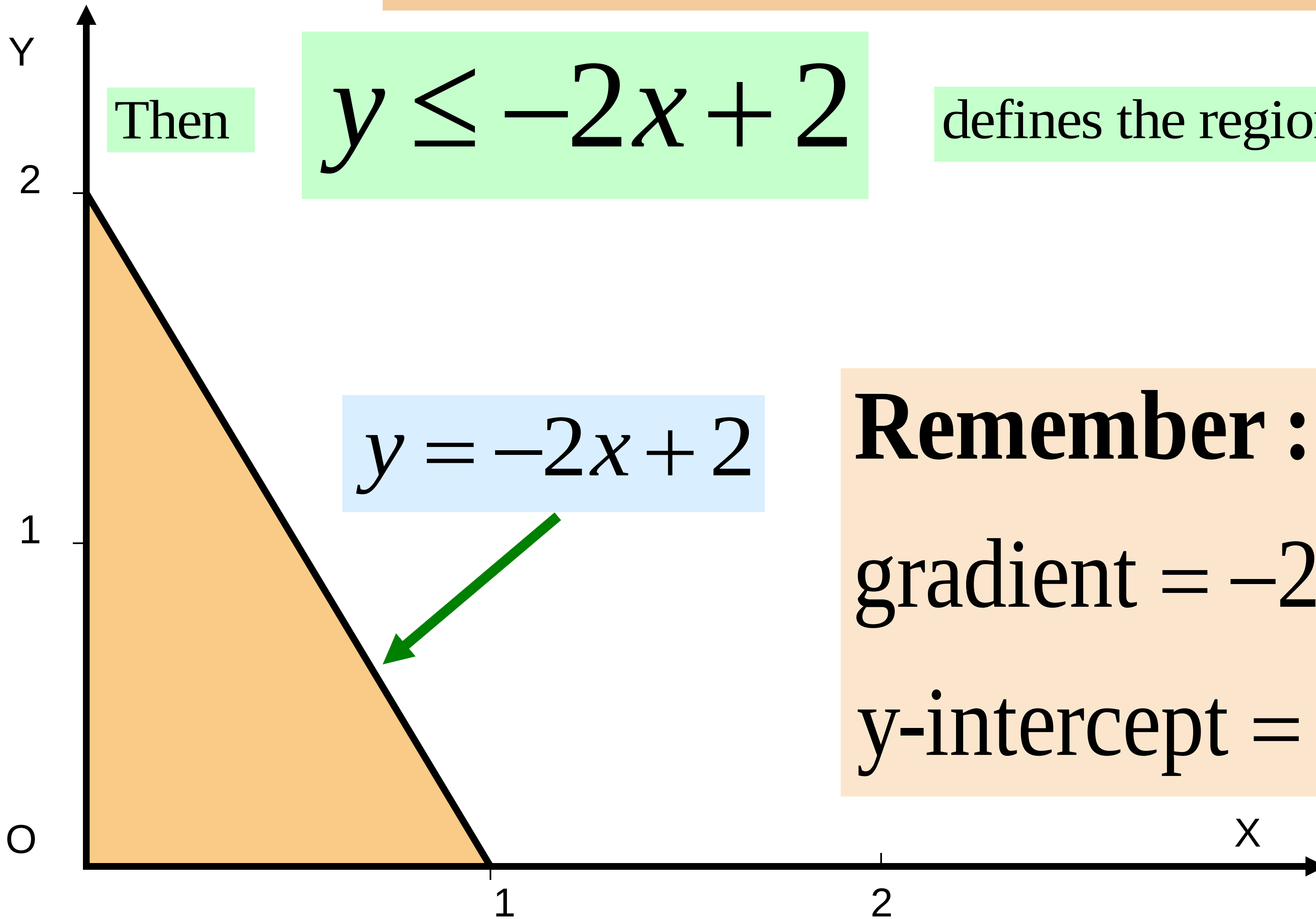
Graphing Linear Inequalities: Example 3



Graphing Linear Inequalities: Example 4

Consider the inequality $2x + y \le 2;\, x \ge 0;\, y \ge 0$

Then $y \le -2x + 2$ defines the region



$y = -2x + 2$

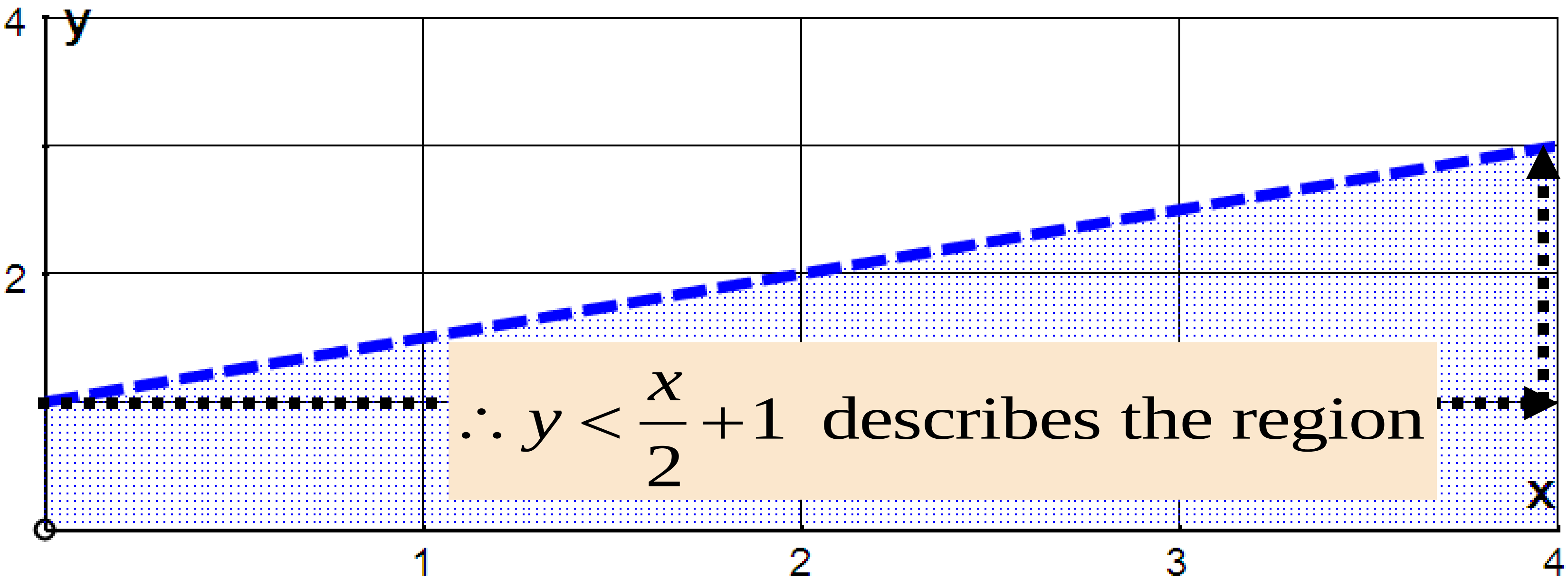
Remember :

gradient $= -2$;

y-intercept $= 2$

Algebraic Description from Graph: Example 6

Given the following graph:



$\therefore\ y < \dfrac{x}{2} + 1$ describes the region

- Determine the defining equation of the line:

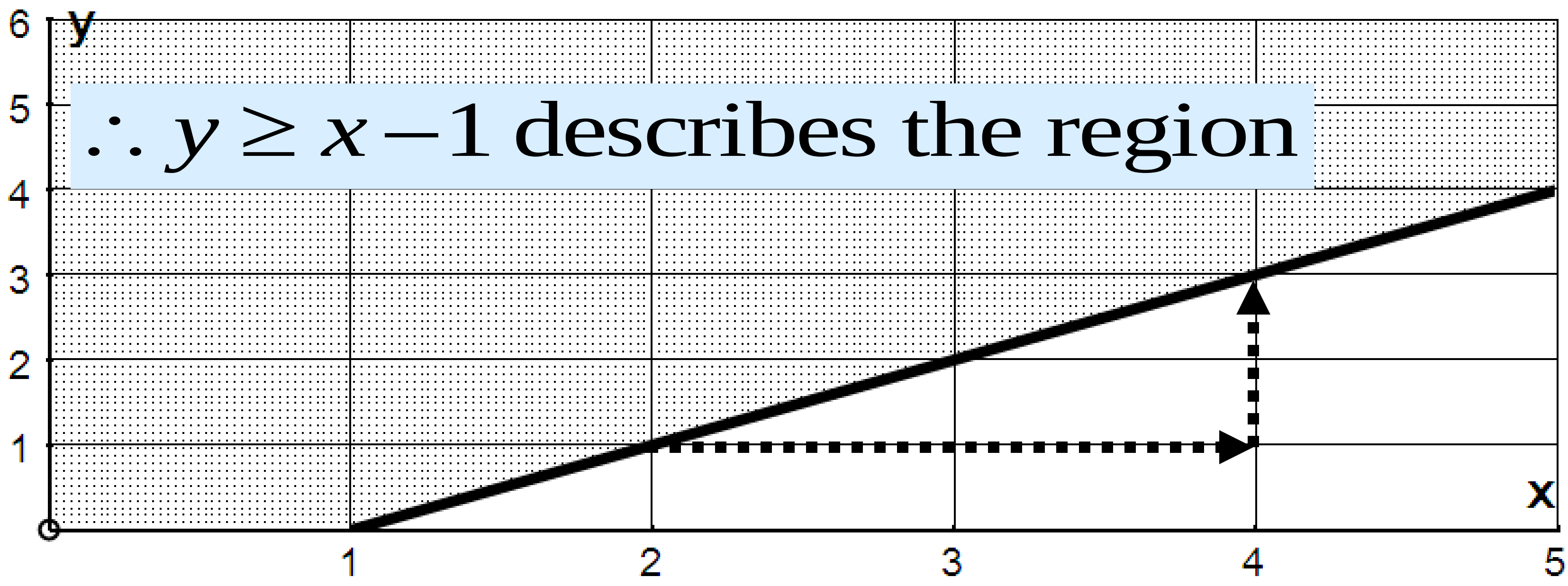
$$m = \frac{\Delta y}{\Delta x} = \frac{2}{4} = \frac{1}{2}$$

and $y-$intercept is 1

$\Rightarrow\ y = mx + c$ or $y = \dfrac{x}{2} + 1$ defines the line

Algebraic Description from Graph: Example 5

Given the following graph:



$\therefore y \geq x - 1$ describes the region

- Determine the defining equation of the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{2}{2} = 1$$

$(1;0)$ lies on this line $\Rightarrow y = 1(x-1) = x - 1$ defines the line

Tutorial 1: Graphing Linear Inequalities

Graph the following sets of inequalities on the Cartesian Plane. Shade the different regions defined by the inequalities.

PAUSE THE DVD

- **Do Tutorial 1**
- **Then View Solutions**

1. $x \geq 0; y \geq 0; x + 2y \geq 12; x + y \geq 8$

2. $x \geq 0; y \geq 0; 4x + 2y \geq 10; x + y < 6$

3. $x \geq 0; y \geq 0; y + 3x \geq 9; x + 4y \geq 12$

Tutorial 1 Problem 1: Suggested Solution

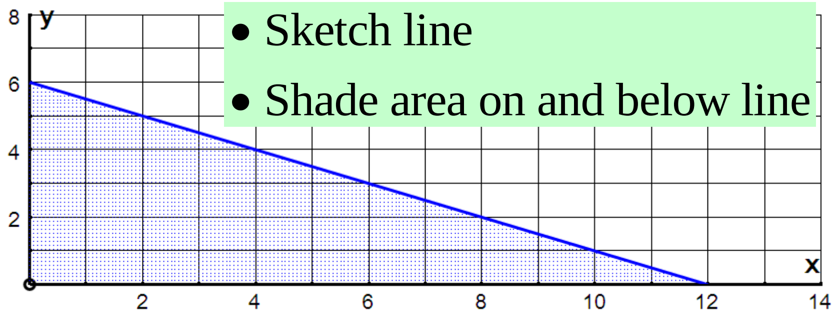
1. Shade the region defined by:

$$x \geq 0; y \geq 0; x + 2y \geq 12; x + y \geq 8$$

$x \geq 0 \wedge y \geq 0$

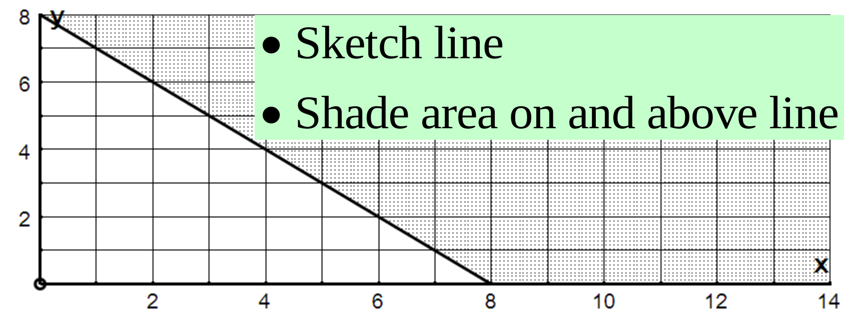
$\Rightarrow$ Restricted to first quadrant

$x + 2y \leq 12 \Rightarrow y \leq -\frac{x}{2} + 6$

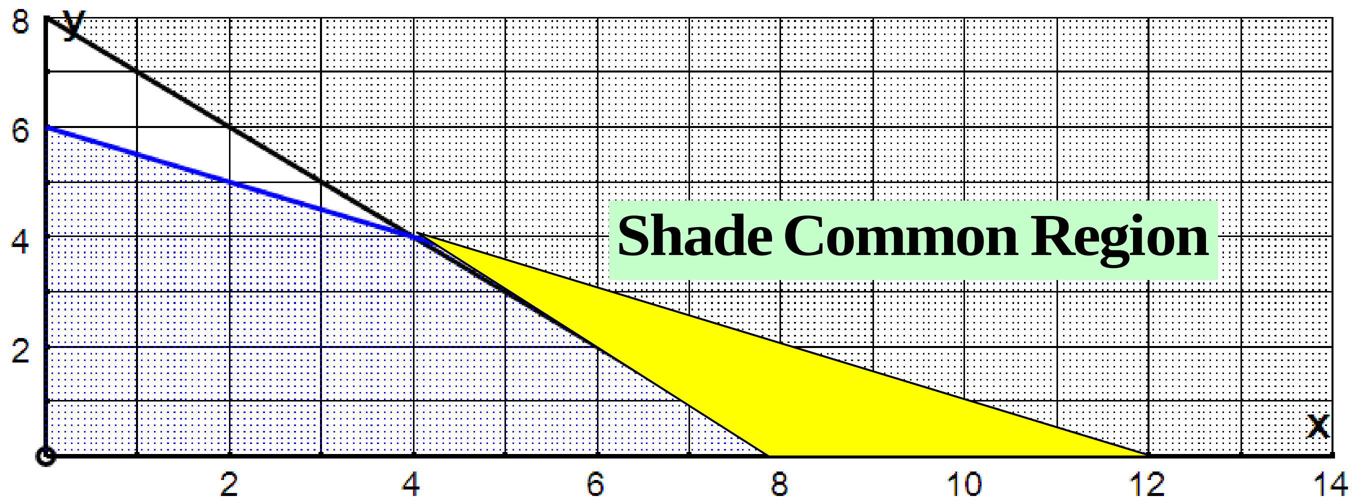


- Sketch line
- Shade area on and below line

$x + y \geq 8 \Rightarrow y \geq -x + 8$



- Sketch line
- Shade area on and above line



Shade Common Region

Tutorial 1 Problem 2: Suggested Solution

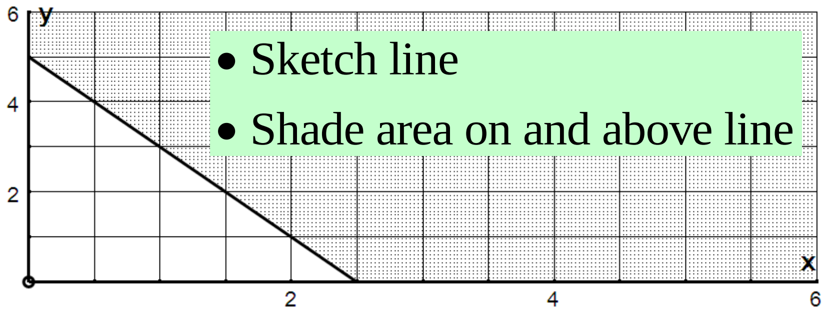
2. Shade region defined by:

$$x \geq 0; y \geq 0; 4x + 2y \geq 10; x + y < 6$$

$x \geq 0 \wedge y \geq 0$

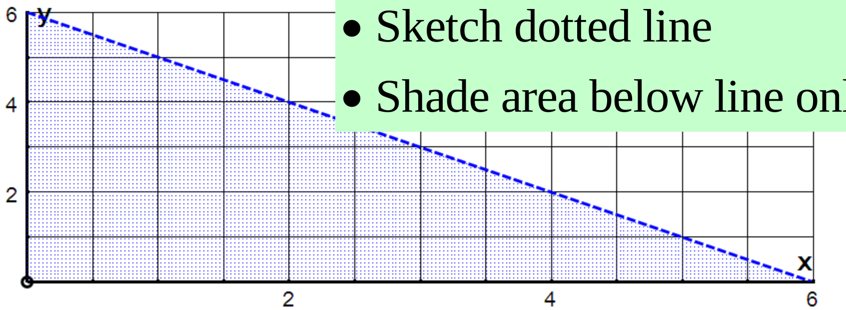
$\Rightarrow$ Restricted to first quadrant

$4x + 2y \geq 10 \Rightarrow y \geq -2x + 5$

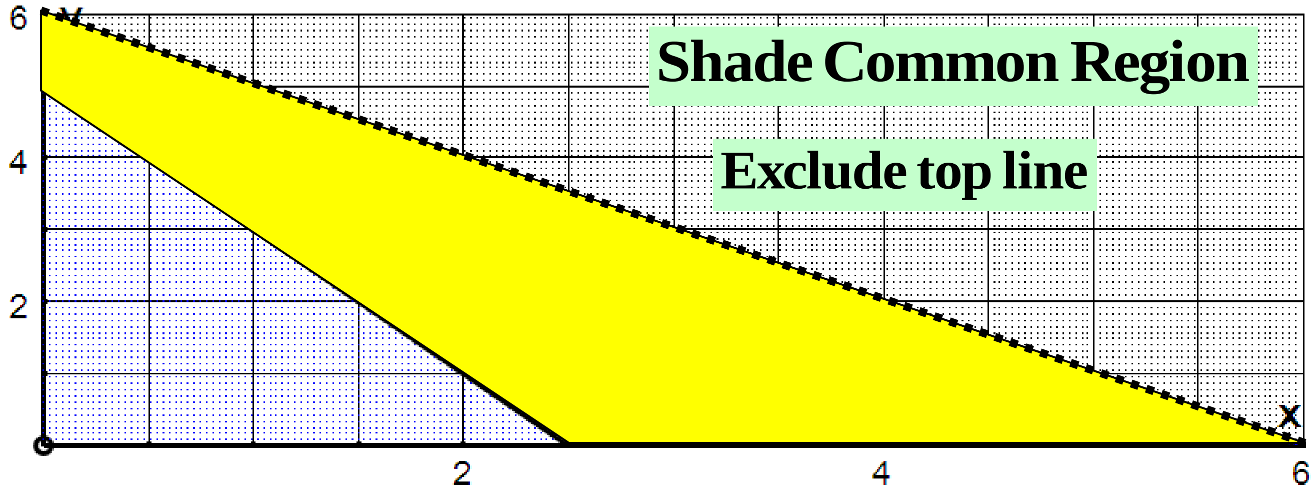


- Sketch line
- Shade area on and above line

$x + y < 6 \Rightarrow y < -x + 6$



- Sketch dotted line
- Shade area below line only



Shade Common Region

Exclude top line

Tutorial 1 Problem 3: Suggested Solution

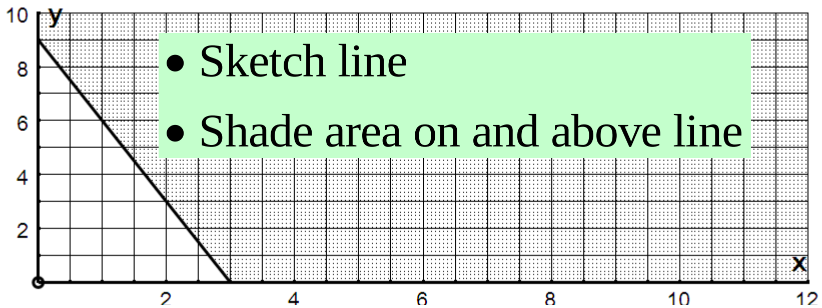
3. Shade region defined by:

$$x \geq 0; y \geq 0; y + 3x \geq 9; x + 4y \geq 12$$

$x \geq 0 \wedge y \geq 0$

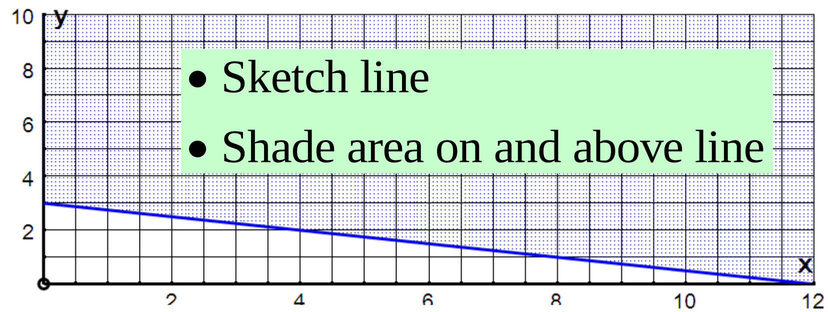
$\Rightarrow$ Restricted to first quadrant

$y + 3x \geq 9 \Rightarrow y \geq -3x + 9$

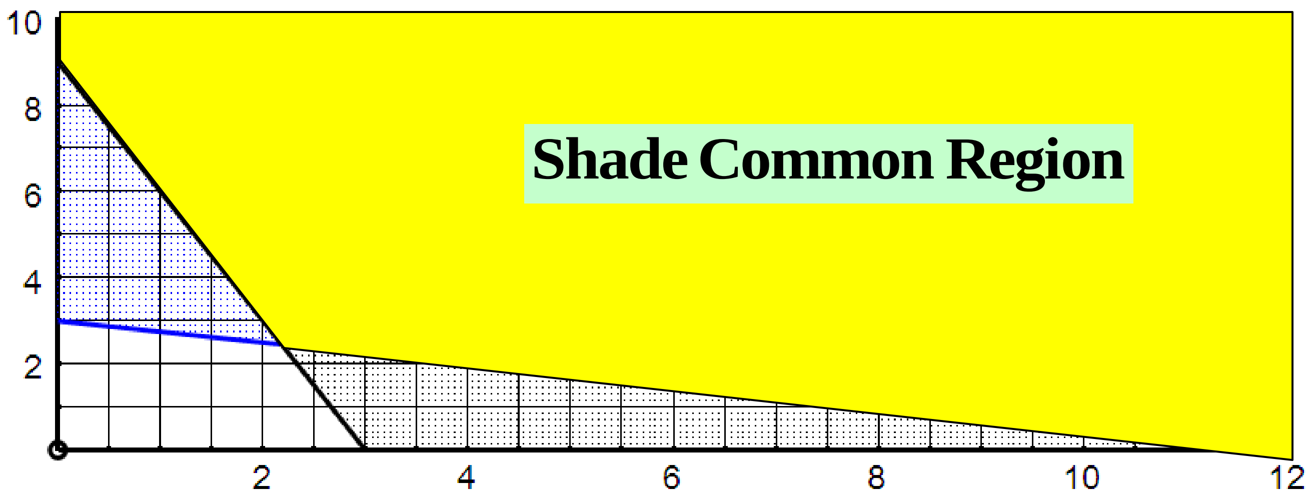


- Sketch line
- Shade area on and above line

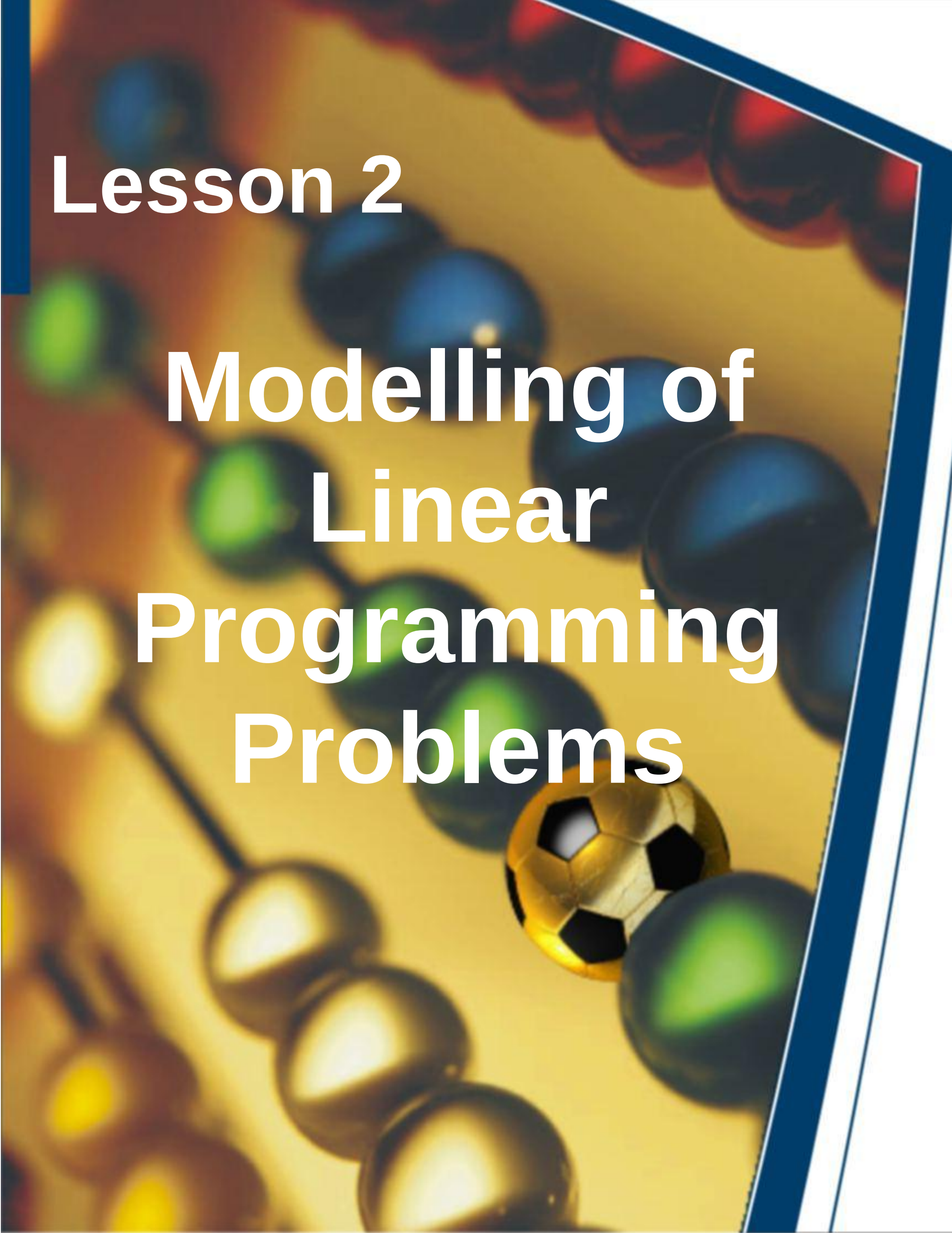
$x + 4y \geq 12 \Rightarrow y \geq -\frac{x}{4} + 3$



- Sketch line
- Shade area on and above line



Shade Common Region



Lesson 2

Modelling of Linear Programming Problems

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Modelling of a Linear Programming Problem

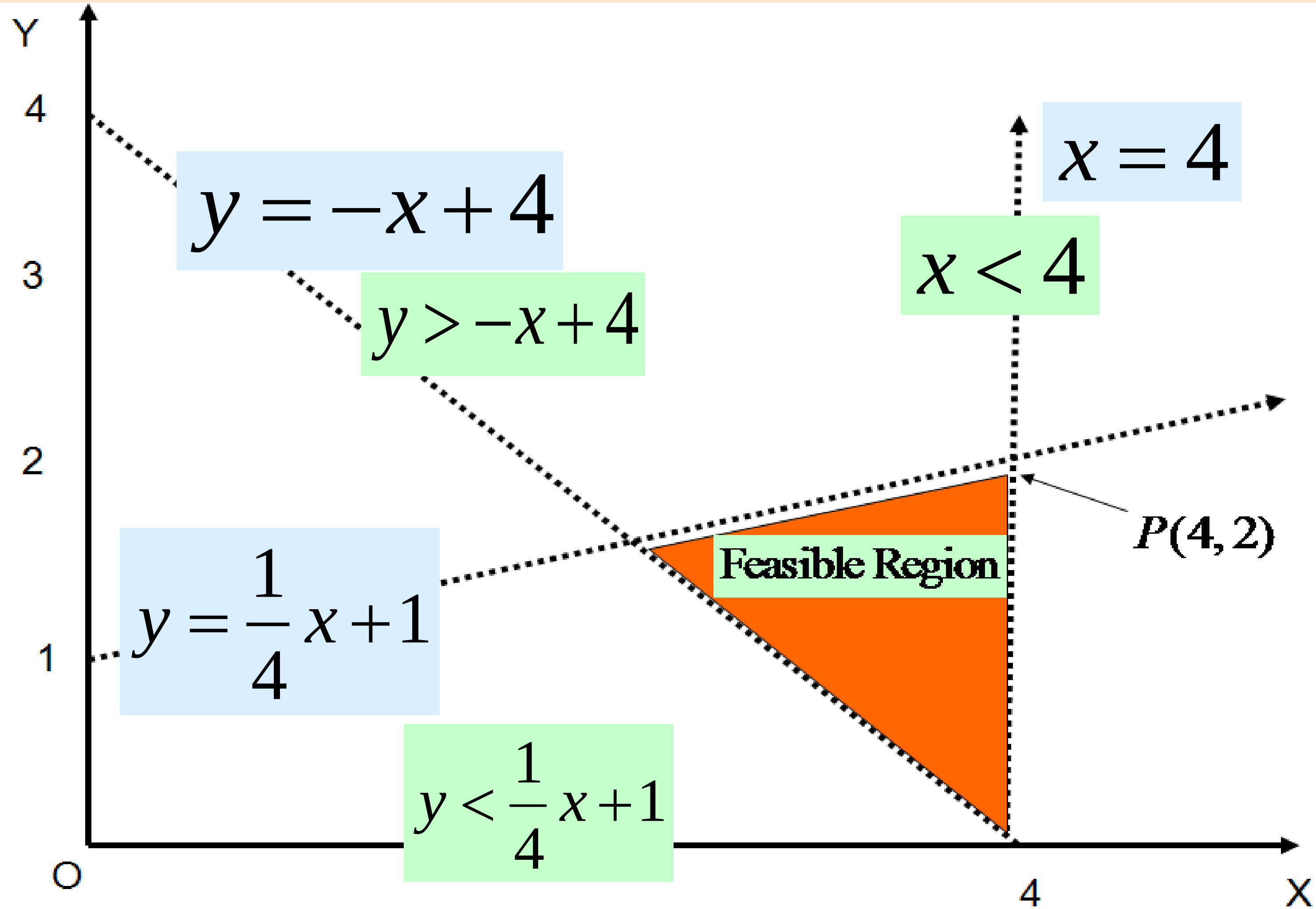
The Mathematical Modelling Phase of a Linear Programming problem involves:

- **Choosing the variables** correctly
- **Formulating the Objective Function** i.t.o. the variables
- **Identifying the linear constraints** as inequalities
- Graphing the constraints to **identify the Feasible Region** in the Cartesian Plane

The Feasible Region in the Cartesian Plane is the set of points within which the most favorable solution of the linear programming problem will lie.

Identify Constraints from Graph

Given the feasible region, find the constraints



Notice the dotted lines resulted in strict inequalities

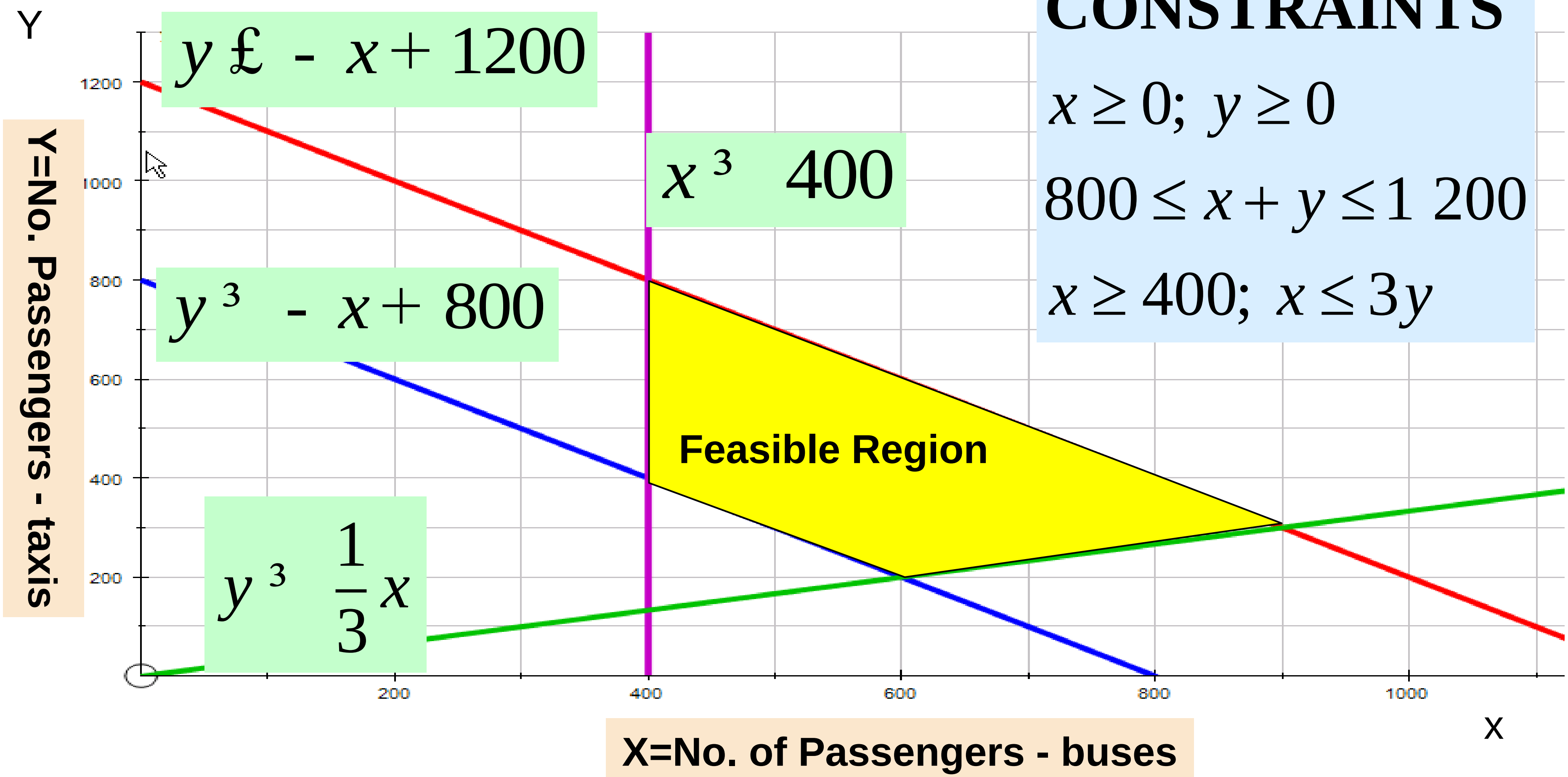
Example 1: Find the Constraints

A company uses buses and minibuses to transport a minimum of 800 and a maximum of 1200 passengers per day. The number of passengers, x, transported by the bus, must be a minimum of 400 per day, but cannot be more than three times the number of passengers, y, transported daily by minibus. (Note that $x, y \in$ □)

$$800 \le x + y \le 1200 \quad x \ge 0 \quad y \ge 0$$

$$x \ge 400 \quad x \le 3y$$

Example 1: Graph Constraints and Identify Feasible Region



CONSTRAINTS

$x \geq 0;\ y \geq 0$

$800 \leq x + y \leq 1\,200$

$x \geq 400;\ x \leq 3y$

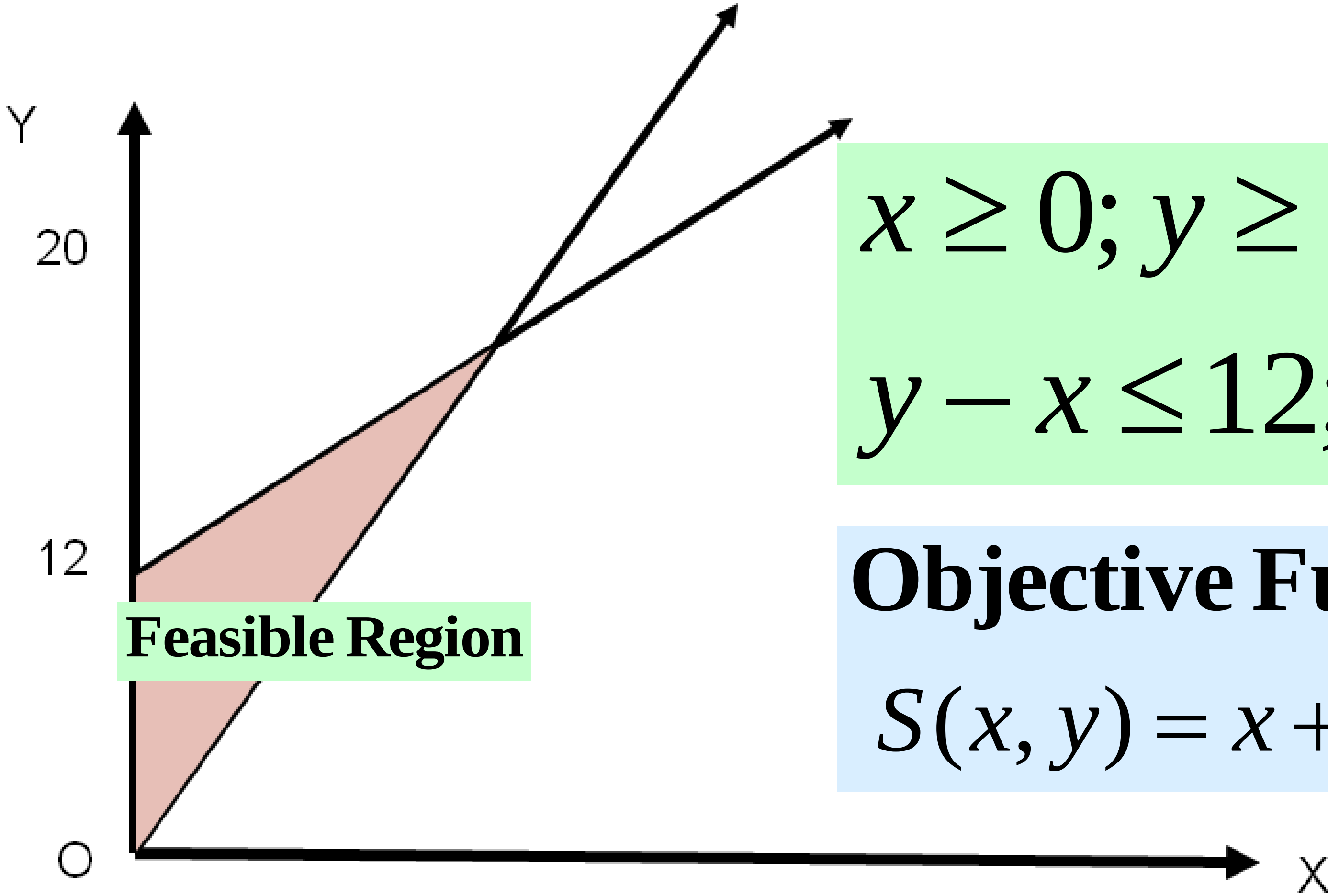
The "best" solution for any objective function always lies within the feasible region

Example 2: Constraints and Feasible Region

The difference between two non-negative integers numbers is at most 12. Maximize the sum if one number is at least four times the other.

Let the numbers be x and y

Constraints :



$$x \geq 0; y \geq 0$$

$$y - x \leq 12; \ y \geq 4x$$

Objective Function :

$$S(x, y) = x + y$$

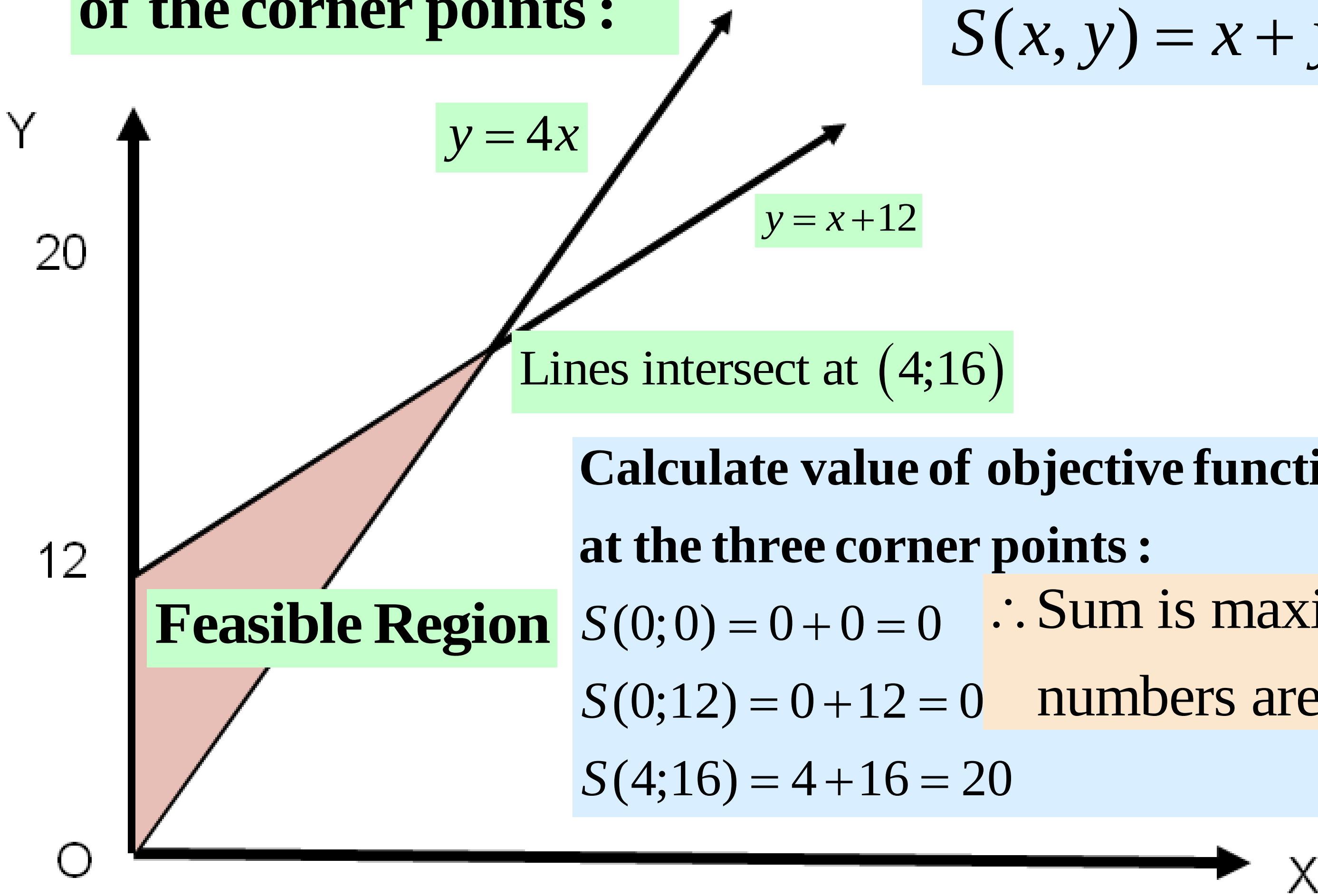
Example 2: Maximize the sum

The difference between two natural numbers is at most 12.
Maximize the sum if one number is at least four times the other.

Determine coordinates of the corner points :

Objective Function :

$$S(x, y) = x + y$$



Calculate value of objective function at the three corner points :

$S(0;0) = 0 + 0 = 0$

$S(0;12) = 0 + 12 = 0$

$S(4;16) = 4 + 16 = 20$

$\therefore$ Sum is maximum when the numbers are 4 and 16

Modelling of a Linear Programming Problem: Example 3

A Factory produces two products A and B. Product A renders a profit of R200 per item and product B renders a profit of R100 per item. The factory **never sells more** of type A than of type B and the sum total of the sales per year **never exceed** 1000 items sold.

How many items of each type must be sold over a year in order to **maximize the profit** from these products?

- Find the variables and the constraints
- Find the objective function
- Graph the feasible region

Example 3: Find Constraints and Objective Function

A Factory produces two products A and B. Product A renders a profit of R200 per item and product B renders a profit of R100 per item. The factory **never sells more** of type A than of type B and the sum total of the sales per year **never exceed** 1000 items sold.

How many items of each type must be sold over a year in order to **maximize the profit** from these products?

$$x + y \leq 1000 \quad x \leq y \quad x \geq 0 \quad y \geq 0$$

Objective

Profit: $P(x;y) = 200x + 100y$

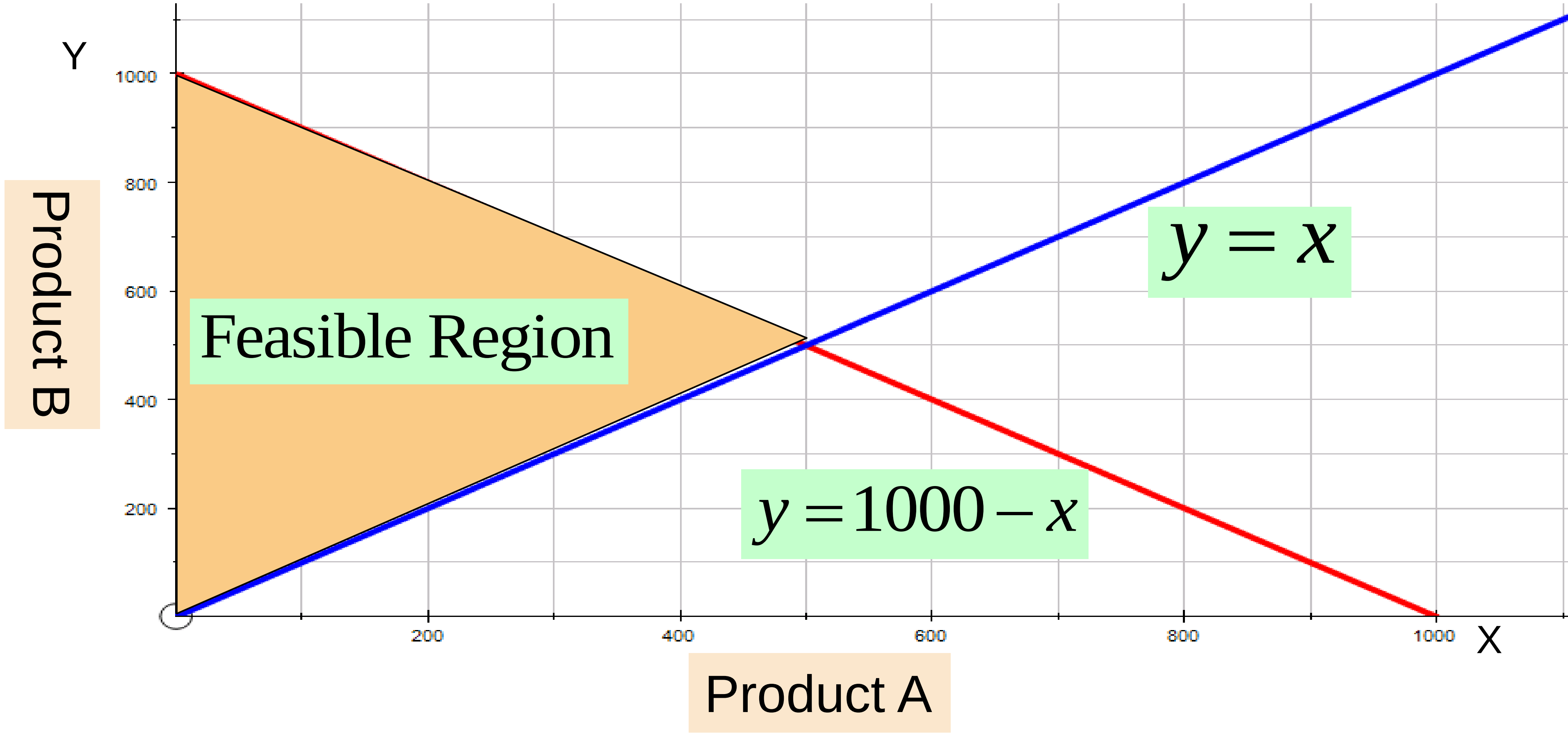
Example 3: Graph Constraints and Identify Feasible Region

Constraints:

$$x + y \le 1000$$

$$\Rightarrow y \le 1000 - x$$

$$y \ge x$$



Example 3: Maximize Profit

Profit: $P(x;y) = 200x + 100y$

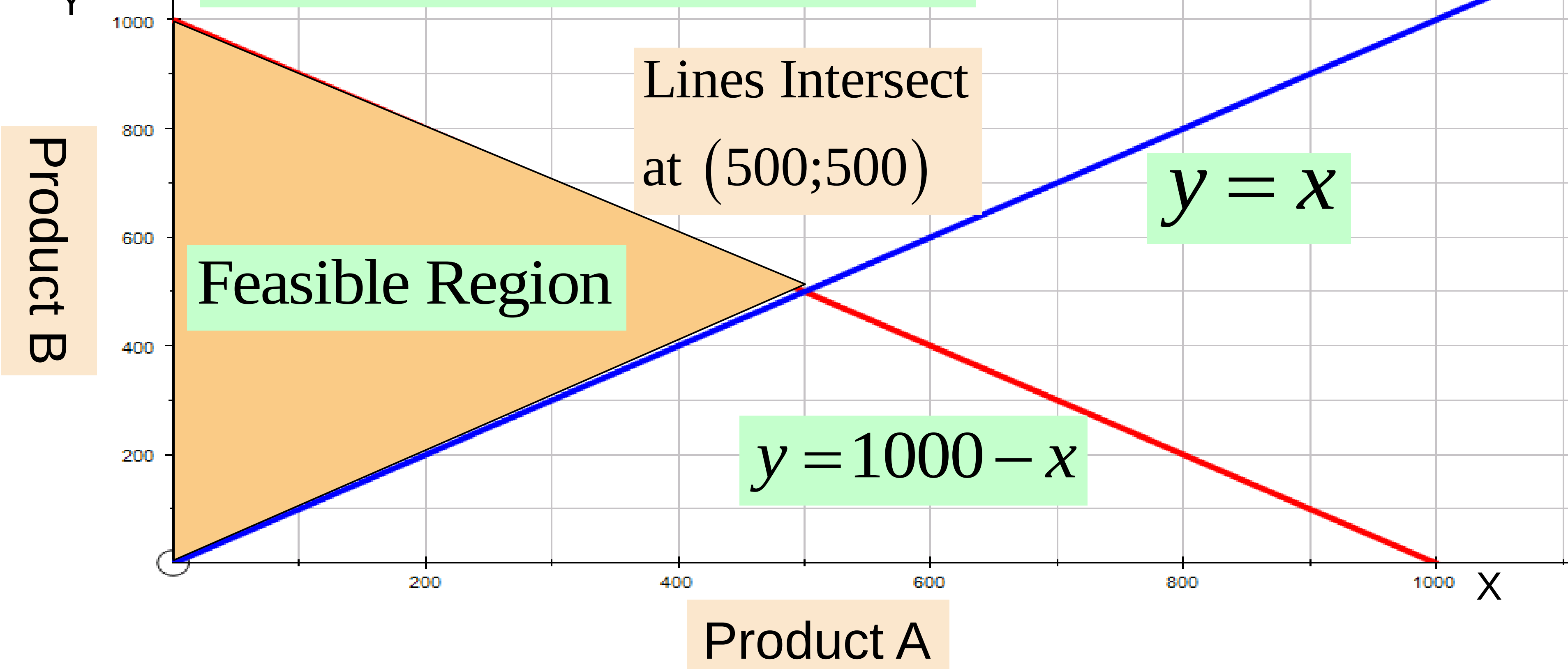
Calculate value of profit at corner points :

$P(0;0) = 200\times 0 + 100\times 0 = 0$

$P(0;1000) = 200\times 0 + 100\times 1\ 000 = 100\ 000$

$P(500;500) = 200\times 500 + 100\times 500 = 150\ 000$

Profit is a maximum if 500 of each product is sold



Example 4: Constraints and Objective Function

Sarah makes bracelets and necklaces to sell at a craft store. Each bracelet makes a profit of $7, takes 1 hour to assemble, and costs $2 for materials. Each necklace makes a profit of $12, takes 2 hours to assemble, and costs $3 for materials. Sarah has 48 hours available to assemble bracelets and necklaces. If she has $78 available to pay for materials, **how many bracelets and necklaces should she make to maximize her profit**?

$x \geq 0$ $y \geq 0$

Cost of materials: $2x + 3y \leq 78$

Time limitation: $x + 2y \leq 48$

Objective Profit: $P(x; y) = 7x + 12y$

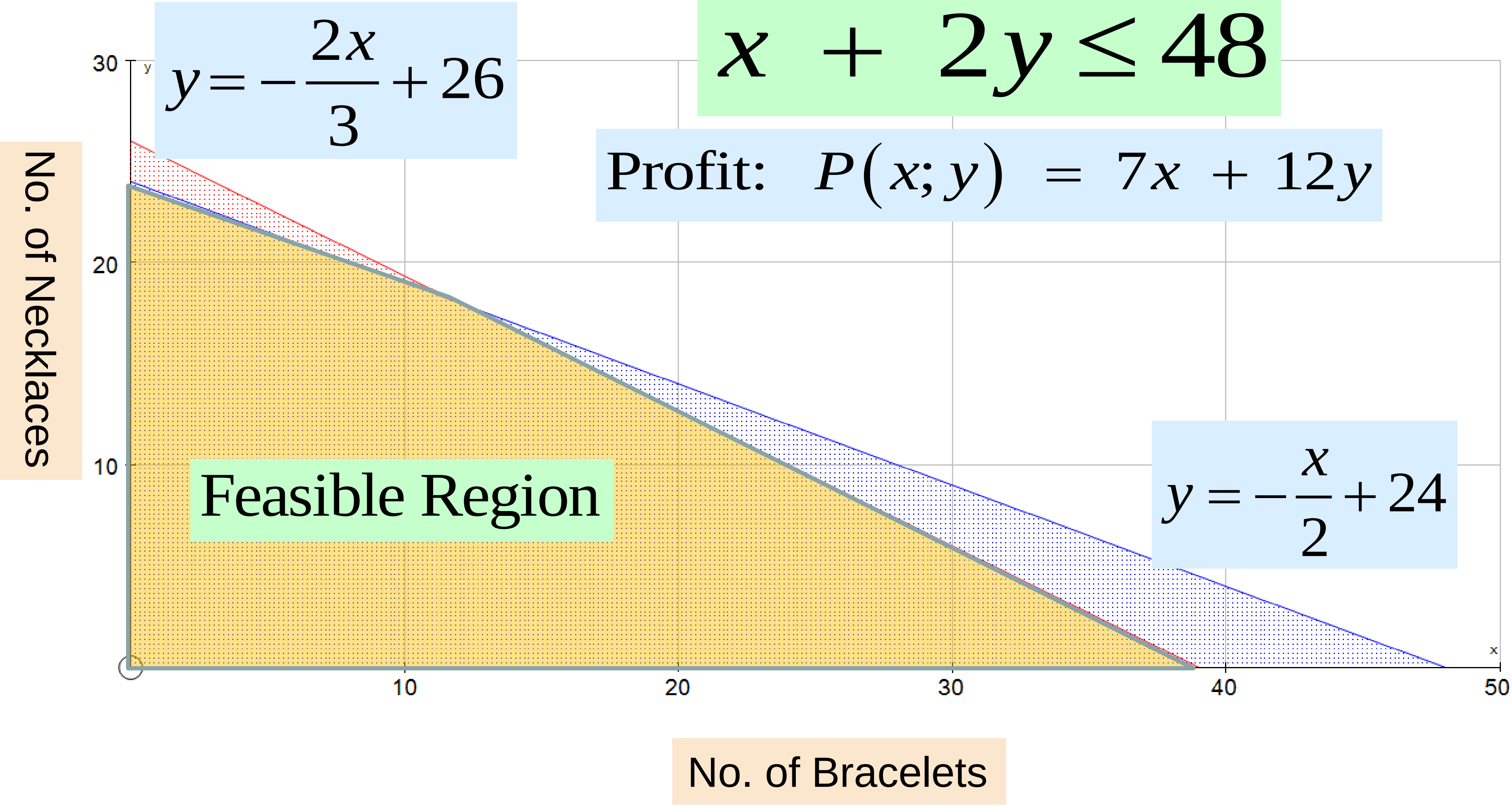
Example 4: Graph Constraints and Identify Feasible Region

Constraints:

$$2x + 3y \le 78$$

$$x + 2y \le 48$$

Profit: $P(x;y) = 7x + 12y$



Example 4: Determine Maximum Profit

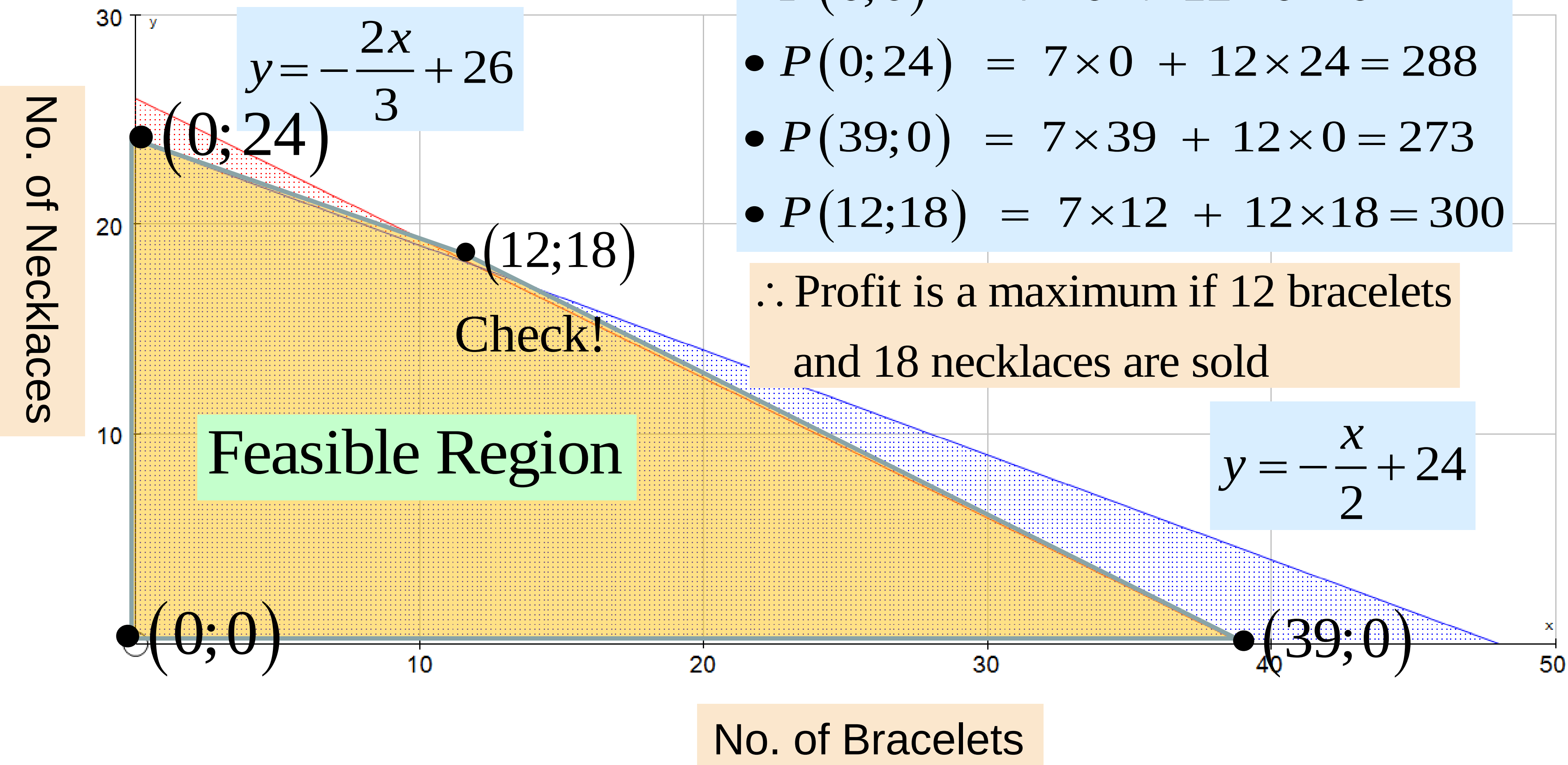
Determine coordinates of corner points.

Profit: $P(x;y) = 7x + 12y$

Calculate Profit at Corner Points:

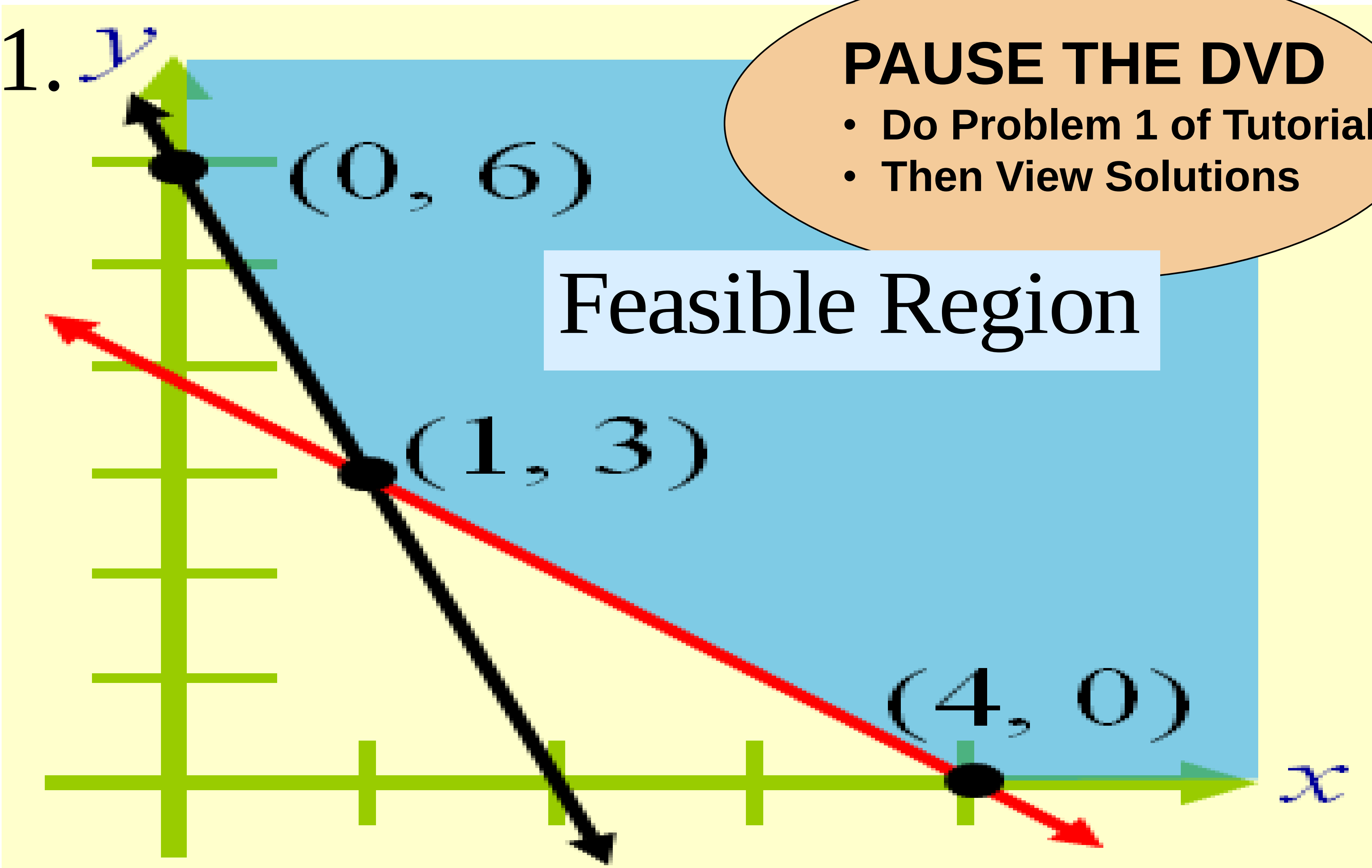
- $P(0;0) = 7\times 0 + 12\times 0 = 0$
- $P(0;24) = 7\times 0 + 12\times 24 = 288$
- $P(39;0) = 7\times 39 + 12\times 0 = 273$
- $P(12;18) = 7\times 12 + 12\times 18 = 300$

$\therefore$ Profit is a maximum if 12 bracelets and 18 necklaces are sold



Given the feasible region, find the constraints

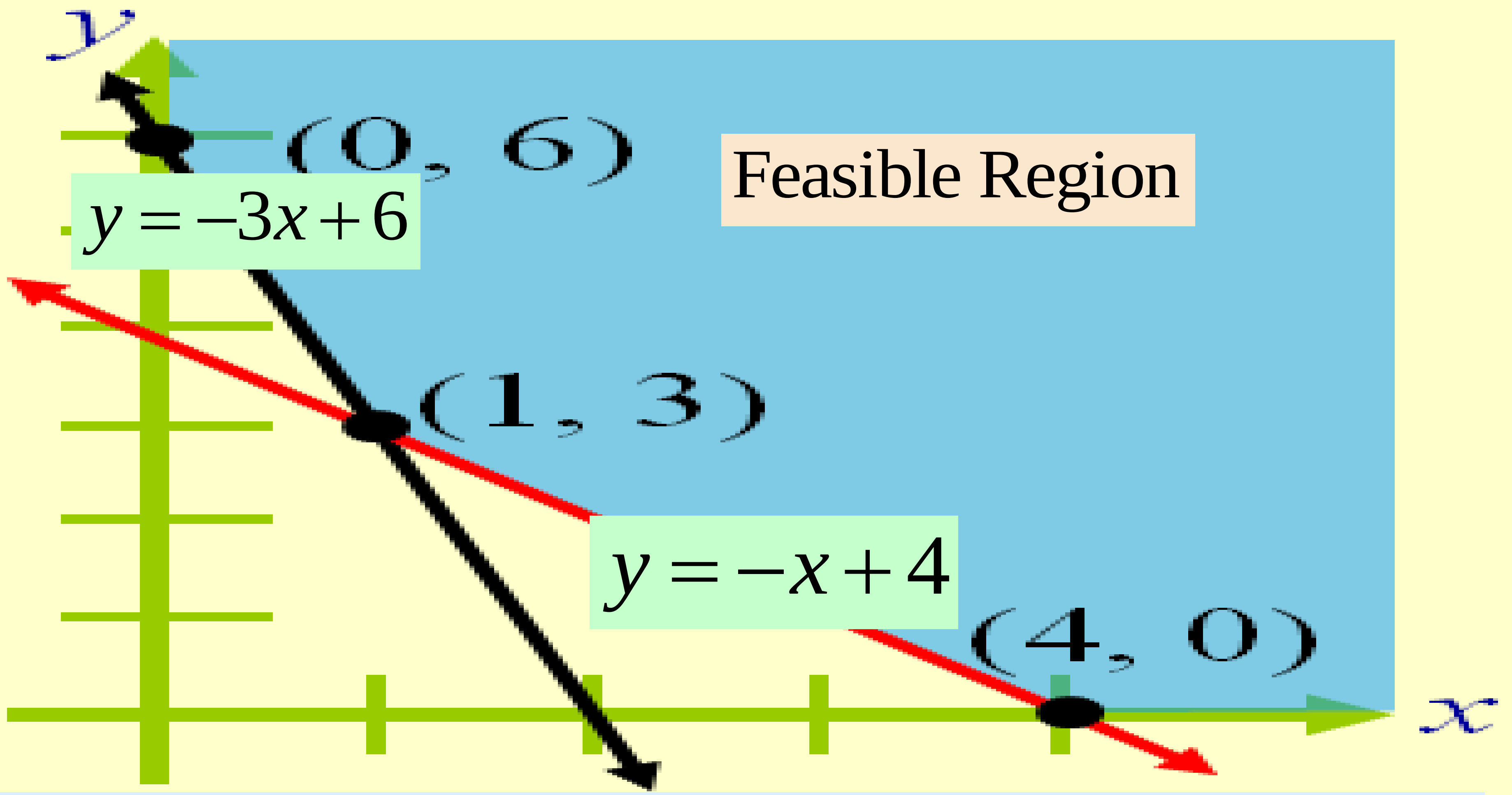
1.



PAUSE THE DVD

- **Do Problem 1 of Tutorial 2**
- **Then View Solutions**

Tutorial 2 Problem 1: Suggested Solution



Constraints :

$$x \geq 0; \; y \geq 0; \quad x + y \geq 4; \quad 3x + y \geq 6$$

Tutorial 2 Problem 2: Constraints, Objective Function and Feasible Region

Identify the constraints and the obj… …ction

in each of the following line… …s.

Also graph the feasible regio…

PAUSE THE DVD

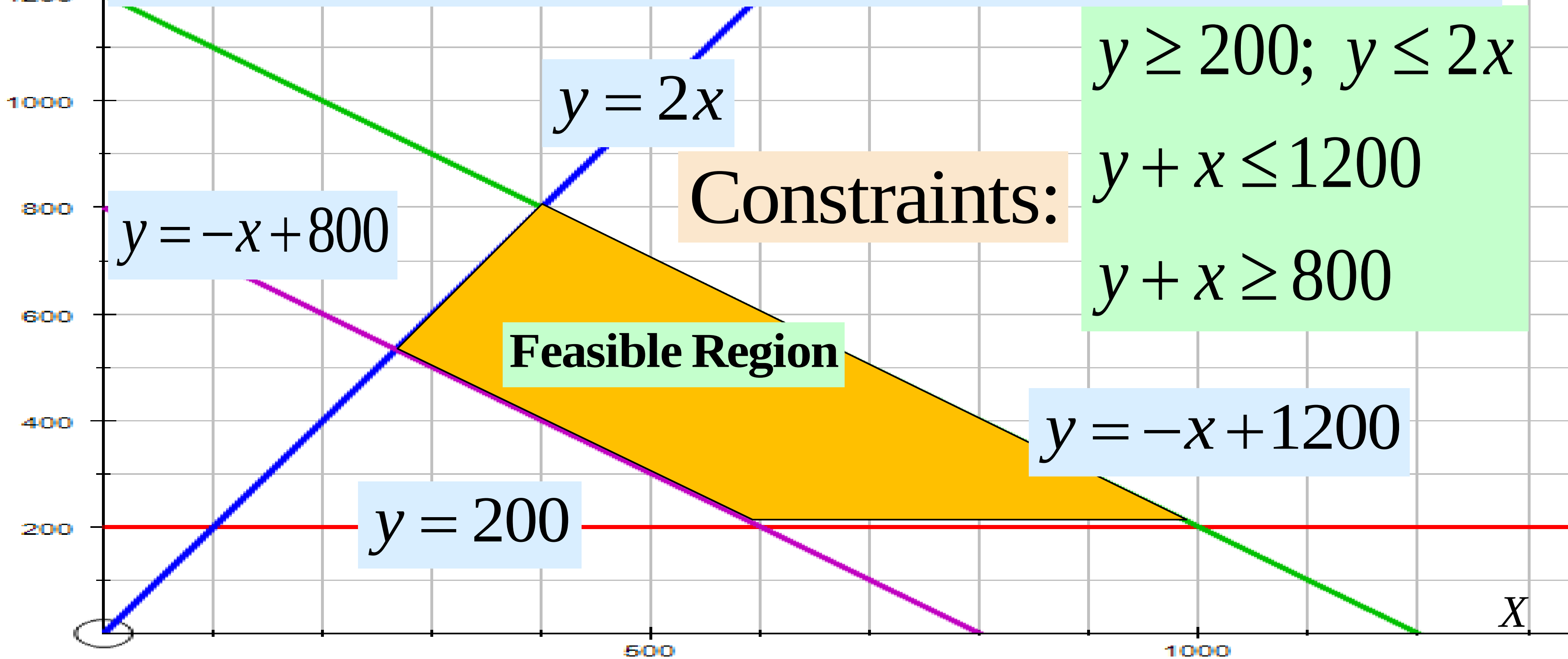
- **Do Problem 2 of Tutorial 2**
- **Then View Solutions**

2. A factory wants to mix two ingredients A and B to form a new product. The mixture must weigh at least 800 kg, but not more than 1200 kg. At least 200 kg of ingredient B must be used and there must be less than 2 kg of ingredient B for every 1 kg of ingredient A in this mixture. Find the most economical mix if ingredient A costs R4/kg and ingredient B costs R7/kg.

Tutorial 2 Problem 2: Constraints, Feasible Region and Objective Function

A factory wants to mix two ingredients A and B to form a new product. The mixture must w… east 800 kg, but not more than 1… At least 200 kg of ingredient … be used and there must less tha… ingredient B for every 1 kg of ingredient A in this mixture. **Find the most economical mix** if ingredient A costs R4/kg and ingredient B costs R7/kg.

x y



Constraints:

$y \geq 200; \; y \leq 2x$

$y + x \leq 1200$

$y + x \geq 800$

Objective Cost function : $C(x;y) = 4x + 7y$

Tutorial 2 Problem 2: Most Economical Mix

Determine coordinates of corner points.

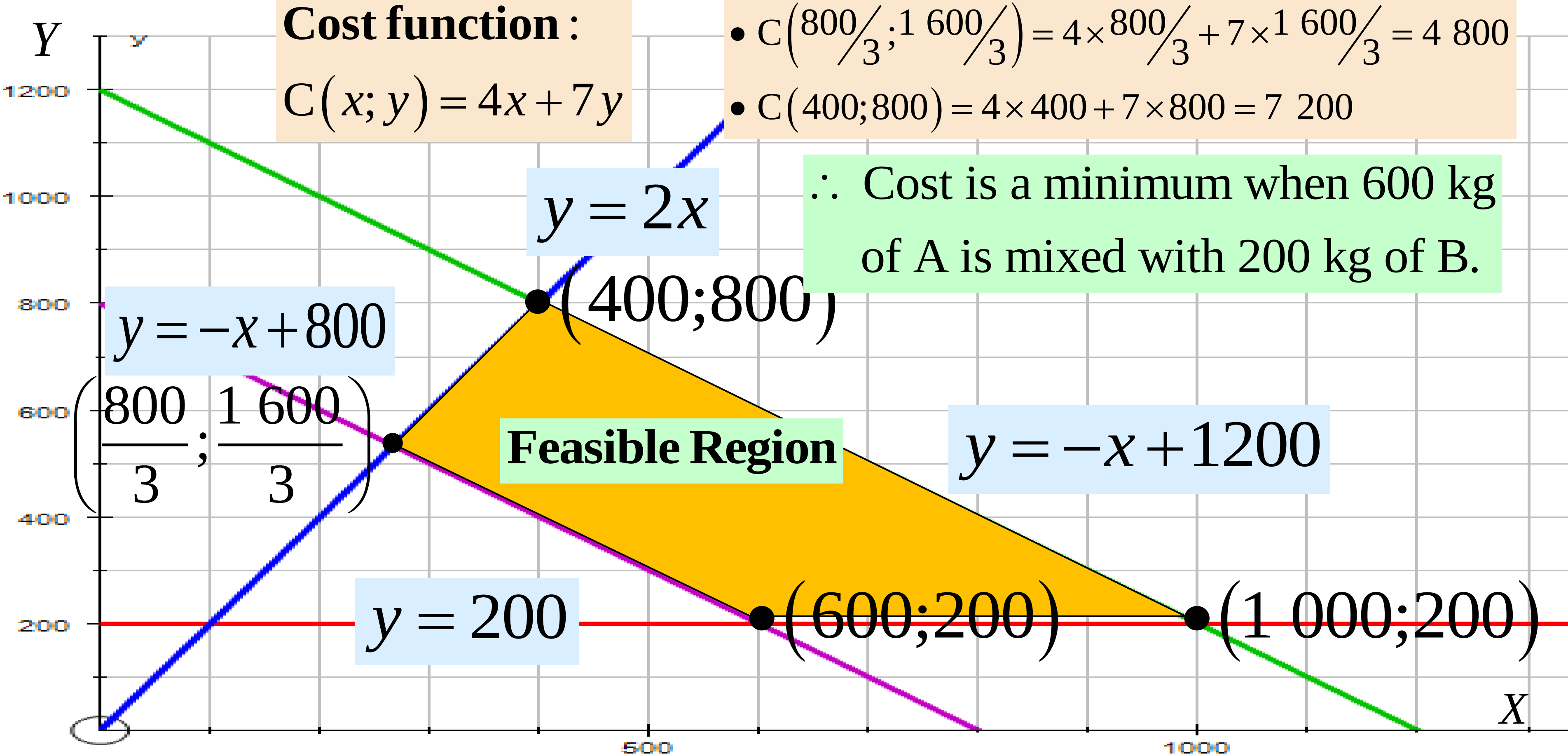
Cost function :

$C(x;y) = 4x + 7y$

Calculate Cost at Corner Points :

- $C(1\ 000;200) = 4\times 1\ 000 + 7\times 200 = 5\ 400$
- $C(600;200) = 4\times 600 + 7\times 200 = 3\ 800$
- $C\left(\frac{800}{3};\frac{1\ 600}{3}\right) = 4\times\frac{800}{3} + 7\times\frac{1\ 600}{3} = 4\ 800$
- $C(400;800) = 4\times 400 + 7\times 800 = 7\ 200$

$\therefore$ Cost is a minimum when 600 kg of A is mixed with 200 kg of B.



3. A farmer has 150 acres of lan… to
grow **potatoes** and **bar…**
uncertainties regardin…
the danger of potato blig…
the potato acreage to 50. Water requirements
are as follows: potatoes need 13.3 cm. per
acre, while barley needs 6.3 cm per acre. The
amount of water available to the farmer in the
coming year is 2,200 acre cm's.
If the farmer estimates the **profit on an acre of potatoes to be R200**, and on an **acre of barley to be R29.**
Determine the **constraints**, **objective function** and identify the **feasible region**.

PAUSE THE DVD

- **Do Problem 3 of Tutorial 2**
- **Then View Solutions**

Tutorial 2 Problem 3: Constraints and Objective Function

- **Identify the unknowns**

x = acreage of potatoes, y = acreage of barley

- **List the Constraints**

$x + y \leq 150$ (Land Limitation)

$x \leq 50$ (Potato Limitation)

$13.3x + 6.3y \leq 2200$ (Water Limitation)

- **State the Objective Function**

$P(x, y) = 200x + 29y$ (To be maximized)

Tutorial 2 Problem 3: Identify Feasible Region

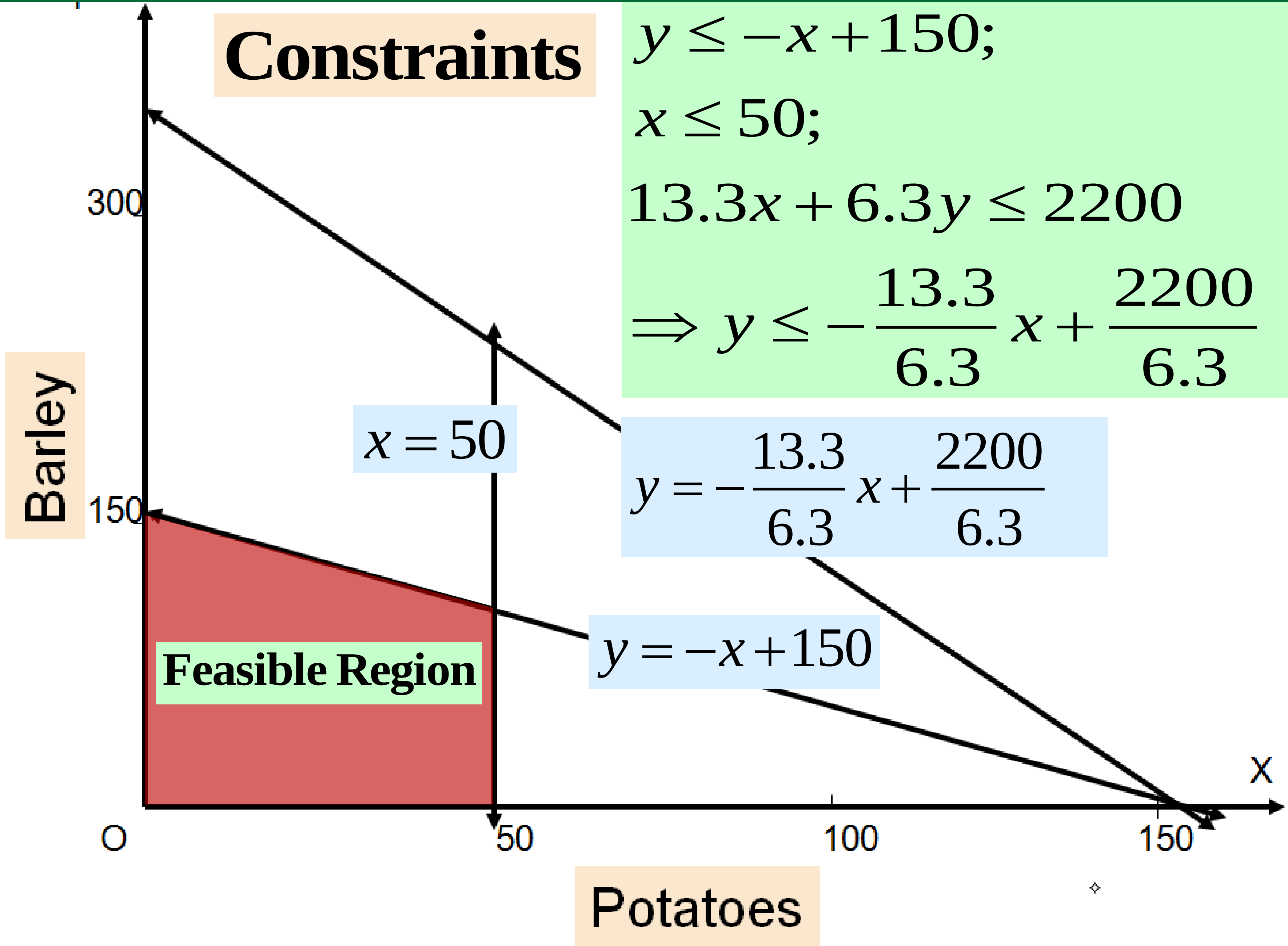
Constraints

$$y \le -x + 150;$$

$$x \le 50;$$

$$13.3x + 6.3y \le 2200$$

$$\Rightarrow y \le -\frac{13.3}{6.3}x + \frac{2200}{6.3}$$



End of the first DVD on Linear Programming

REMEMBER!

- Consult text-books for additional examples.
- Attempt as many as possible other similar examples on your own.
- Compare your methods with those that were discussed in the DVD.
- Repeat this procedure until you are confident.
- Do not forget:

Practice makes perfect!